Final Math 142 Fall 2001

PRINT Your Name:____________________

Get your course grade from **TIPS/VIP** late on Tuesday or later. There are 20 problems on 10 pages. The exam is worth 150 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

1 (8 points) Find $\int \sin^5 x\,dx$ (Be sure to check your answer.)

integral $= \int (1-\cos^2 x)^2 \sin x\,dx = -\int (1-u^2)^2\,du = -\int (1-2u^2+u^4)\,du$

$u = \cos x$

$du = -\sin x$

$$-\left(\cos x - 2\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5}\right) + C$$

✓ $\frac{d}{dx}(P.A.) = \left(-\sin x + 2\cos^2 x \sin x - \cos^4 x \sin x\right)$

$\sin x\,(1 - 2\cos^2 x + \cos^4 x)$

$= \sin x\,(1-\cos^2 x)^2$ ✓

2. (8 points) Find $\int \sin^4 x\,dx = \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\,dx$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\,dx$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + C$$

3. (8 points) Where does the function $f(x) = \sum_{n=1}^{\infty} \frac{(x - 7)^n}{n \cdot 3^n}$ converge? **Justify** your answer.

$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-7|^{n+1}}{n+1 \cdot 3^{n+1}} \frac{n \cdot 3^n}{|x-7|^n}$

$= \lim_{n\to\infty} \frac{n}{n+1} \frac{|x-7|}{3} = \frac{|x-7|}{3}$

If $\rho < 1$, the $f(x)$ converses

If $1 < \rho$, then $f(x)$ diverges

$\rho < 1$ when $\frac{|x-7|}{3} < 1$

$|x-7| < 3$

$-3 < x-7 < 3$

$4 < x < 10$

at 4 $f(4) = \sum \frac{(-3)^n}{n 3^n} = \sum \frac{(-1)^n}{n}$

This is the alternating Harm. series. It converges

at 10 $f(10) = \sum \frac{3^n}{n 3^n} = \sum \frac{1}{n}$

This is the Harmonic series. It diverges

$f(x)$ converges for

$4 \leq x < 10$

$f(x)$ diverges elsewhere else.

4. (8 points) Solve the differential equation $\frac{dy}{dt} = 6y$ with the initial condition $y(1) = 4$. Check your answer.

$\int \frac{dy}{y} = \int 6\,dt$

$\ln y = 6t + c$

$e^{\ln y} = e^{6t} e^c$

$y = Ke^{6t}$

$4 = Ke^6$

$\frac{4}{e^6} = K$

$y = \frac{4}{e^6} e^{6t}$

Check $y(1) = \frac{4}{e^6} e^6 = 4$ ✓

$\frac{dy}{dt} = 6\frac{4}{e^6} e^{6t} = 6y$ ✓

5. (8 points) Does $\sum_{n=1}^{\infty} 1-\frac{2}{n}^{\;n}$ converge? Justify your answer.

The limit of the n^{th} term is $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{n} = e^{-2}$

This limit is not zero. Thus $\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^{n}$ diverges by the n^{th} term test.

The series "Telescopi

6. (8 points) Does $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ converge? Justify your answer.

The m^{th} partial sum is $s_m=\sum_{n=1}^{m} \frac{1}{n}-\frac{1}{n+1} = \left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cdots\right)+\cdots+\left(\cancel{\frac{1}{m}}-\frac{1}{m+1}\right)$

$1-\frac{1}{m+1}$

The sum of a series is the limit of the sequence of partia sums. The sum $\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1} = \lim_{m\to\infty} s_m = \lim_{m\to\infty} 1-\frac{1}{m+1} = 1$

Thus $\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}$ converges to 1.

7. (8 points) Does $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge? Justify your answer

test $\rho = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \lim_{n \to \infty} \frac{n+1}{n} \frac{1}{2} = \frac{1}{2} <$

$\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by the ratio

8. (8 points) Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^5}$ converge? Justify your answer.

Compare to $\sum \frac{1}{n^4}$ which is the p-series with $p = 4 > 1$ so $\sum \frac{1}{n^4}$

converges. $\ln n < n$ so $\frac{\ln n}{n^5} < \frac{n}{n^5} = \frac{1}{n^4}$

$\sum_{n=1}^{\infty} \frac{\ln n}{n^5}$ also converges

9. (8 points) Suppose that the government pumps an extra \$1 billion into the economy. Assume that each business and individual saves 25% of its income and spends the rest, so that of the initial \$1 billion, 75% is respent by individuals and businesses. Of that amount, 75% is spent, and so forth. What is the total increase in spending due to the government action?

Extra spending $= B + \frac{3}{4}B + \left(\frac{3}{4}\right)^2 B + \left(\frac{3}{4}\right)^3 B + \cdots = \frac{B}{1-\frac{3}{4}} =$ \$4 billion

where $B =$ \$1 Billion

Geometric series
ratio $\frac{3}{4}$
$a = B$

10. (8 points) Find the general solution of $\frac{dy}{dx} - \frac{y}{x} = 3x^3$. Check your answer.

This problem has the form $y' + P(x)y = Q(x)$

Multiply both sides by $\mu = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$\frac{1}{x}y' - \frac{y}{x^2} = 3x^2$

The left side is $\frac{d}{dx}\left(\frac{1}{x}y\right)$

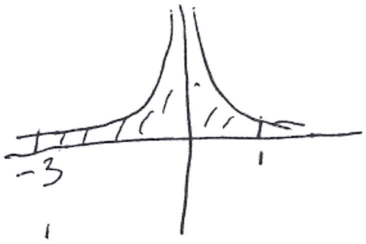
So $\frac{1}{x}y = \int 3x^2 dx = x^3 + c$

$y = x^4 + cx$

$\frac{dy}{dx} - \frac{y}{x} = 4x^3 + c - \frac{x^4 + cx}{x}$

$= 3x^3$

11. (7 points) Find $\displaystyle\int_{-3}^{1} \frac{1}{x^2}\,dx$.



integral $= \displaystyle\lim_{a\to 0^-} \int_{-3}^{a} \frac{1}{x^2}dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x^2}dx$

$= \displaystyle\lim_{a\to 0^-} -\frac{1}{x}\Big]_{-3}^{a} + \lim_{b\to 0^+} -\frac{1}{x}\Big]_{b}^{1}$

$= \displaystyle\lim_{a\to 0^-} -\frac{1}{a} - \frac{1}{3} + \lim_{b\to 0^+} + \frac{1}{b}$

$= +\infty - \frac{1}{3} - 1 + \infty$

The integral diverges to $+\infty$.

12. (7 points) Find $\displaystyle\int \sqrt{1+x^2}\,dx$. (Be sure to check your answer.)

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$1+x^2 = 1+\tan^2\theta = \sec^2\theta$

Integral $= \displaystyle\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$

$u = \sec\theta \qquad v = \tan\theta$

$du = \sec\theta\tan\theta\, d\theta \quad dv = \sec^2\theta\, d\theta$

so $\displaystyle\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$

$2\displaystyle\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$

$\displaystyle\int \sec^3\theta\, d\theta = \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big] + C$

ans $= \dfrac{1}{2}\Big[\sqrt{1+x^2}\,x + \ln\big|\sqrt{1+x^2} + x\big|\Big] + C$

✓: $\frac{d}{dx}$?A

$= \dfrac{1}{2}\left[\dfrac{x^2}{\sqrt{1+x^2}} + \sqrt{1+x^2} + \dfrac{\frac{x}{\sqrt{1+x^2}} + 1}{\sqrt{1+x^2} + x}\right]$

$\dfrac{1}{2}\left[\dfrac{x^2 + 1 + x^2}{\sqrt{1+x^2}} + \dfrac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}\,(\sqrt{1+x^2} + x)}\right]$

$= \dfrac{1}{2}\left[\dfrac{2x^2+1}{\sqrt{1+x^2}} + \dfrac{1}{\sqrt{1+x^2}}\right]$

$= \dfrac{x^2+1}{\sqrt{1+x^2}} = \sqrt{1+x^2}$ ✓

13. (7 points) Find $\int \frac{x}{\sqrt{9-16x^2}}\,dx$ (Be sure to check your answer.)

$u = 9 - 16x^2$

$du = -32x\,dx$

integral $= \frac{-1}{32}\int u^{-\frac{1}{2}}\,du = \frac{-1}{32}\, 2u^{\frac{1}{2}} + c = \frac{-1}{16}\sqrt{9-16x^2} + c$

$\frac{d}{dx}(PA) = \frac{\frac{-1}{16}(-16x)2}{2\sqrt{9-16x^2}}$ ✓

14. (7 points) Find $\int \frac{1}{\sqrt{9-16x^2}}\,dx$. (Be sure to check your answer.)

$u = 4x$

$du = 4dx$

integral $= \frac{1}{4}\int \frac{du}{\sqrt{9}} = \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + c$

$\int \frac{du}{\sqrt{9^2-u^2}} = \sin^{-1}\frac{u}{9} + c$

✓: $\frac{d}{dx}(PA) = \frac{1}{4}\,\frac{\frac{4}{3}}{\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \frac{1}{\sqrt{9\left(1-\frac{16x^2}{9}\right)}}$ ✓

15. (7 points) Find $\int \ln x \, dx$. (Be sure to check your answer.)

$u = \ln x \quad v = x$

$du = \frac{1}{x}dx \quad dv = dx$

$x \ln x - \int dx = x \ln x - x + C$

$\checkmark$: $\frac{d}{dx}(\text{ans}) = x\frac{1}{x} + \ln x - \;\; = \ln x \checkmark$

16. (7 points) Graph $r = 2 - 4\sin\theta$. Find the area inside the inner loop of this graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
r	2	0	-2	0	2	6	2

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} (2-4\sin\theta)^2 \, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 - 16\sin\theta + \frac{16}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(12\theta + 16\cos\theta - 8\,\frac{\sin 2\theta}{2}\right)\Big]_{\pi/6}^{5\pi/6}$$

$$\frac{1}{2}\left(12\left(\frac{4\pi}{6}\right) - 32\frac{\sqrt{3}}{2} + 8\frac{\sqrt{3}}{2}\right)$$

$$\frac{8\pi - 12\sqrt{3}}{2}$$

17. (7 points) Let $f(x) = x \ln x$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of $y = f(x)$. Graph $y = f(x)$.

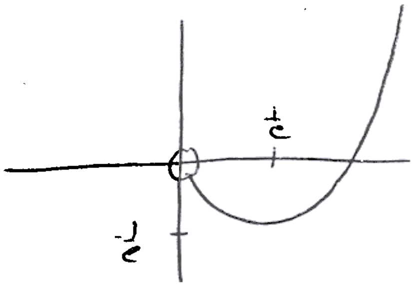
$f'(x) = x \cdot \frac{1}{x} + \ln x$

$= 1 + \ln x$ f' neg | f' pos at $\frac{1}{e}$

$f' = 0$ at $x = \frac{1}{e}$

$f'' = \frac{1}{x}$ f'' always positive

Note the domain is $0 < x$



$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim_{x \to 0} -x = 0$

f is inc for $\frac{1}{e} < x$.

f is dec for $0 < x < \frac{1}{e}$

f is always concave up

$(\frac{1}{e}, -\frac{1}{e})$ is the local min

no local max

no point of infl.

18. (7 points) Find $\displaystyle\int \frac{3 + 2x + 7x^2 - 3x^3}{x^4 + x^2}\, dx$. (Be sure to check your answer.)

$\frac{3 + 2x + 7x^2 - 3x^3}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

ans $= 2\ln|x| - \frac{3}{x} - \frac{5}{2}\ln(x^2+1) + 4\tan^{-1}x + C$

$3 + 2x + 7x^2 - 3x^3 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$

$= Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$

$3 = B$

$A = 2$

$7 = B + D$ so $D = 4$

$-3 = A + C$ so $C = -5$

Check $\frac{2}{x} + \frac{3}{x^2} + \frac{-5x+4}{x^2+1}$

$= \frac{2x(x^2+1) + 3(x^2+1) + (-5x+4)x^2}{x^2(x^2+1)}$

$\frac{-3x^3 + 7x^2 + 2x + 3}{x^2(x^2+1)}$

19. (7 points) Which familiar function is equal to $\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\frac{x^3}{5!}+\frac{x^4}{6!}+\frac{x^5}{7!}+$ Justify your answer.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^x - 1 - x = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} +$$

20. (7 points) Find the third Taylor polynomial $P_3(x)$ for $f(x) = \ln x$ about $a = 1$. Estimate the error that is introduced if $f(x)$ is approximated by $P_3(x)$ for $.8 \le x \le 1.2$.

$f(x) = \ln x \quad f(1) = 0$

$f'(x) = \frac{1}{x} \quad f'(1) =$

$f''(x) = \frac{-1}{x^2} \quad f''(1) = -\frac{1}{1}$

$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$

$f^{(4)}(x) = \frac{-6}{x^4}$

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$$|f(x) - P_3(x)| = |R_3(x)| \le \left|\frac{f^{(4)}(c)(x-1)^4}{4!}\right| = \left|\frac{6}{24c^4}\right| |x-1$$

for some c between x and 1.

$$\le \frac{1}{4(.8)^4}(.2)^4$$

$\frac{1}{c}$ is largest when c is smallest

$.8 \le x \le 1.2 \Rightarrow -.2 \le x-1 \le .2$

So $|x-1| \le .2$

So $|f(x) - P_3(x)| \le \frac{1}{4(4)^4}$