

Final Math 142 Fall 2001

PRINT Your Name: _____

Get your course grade from **TIPS/VIP** late on Tuesday or later. There are 20 problems on 10 pages. The exam is worth 150 points. **SHOW** your work. **CIRCLE** your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

1 (8 points) Find $\int \sin^5 x \, dx$ (Be sure to check your answer.)

$$\text{integral} = \int (1 - \cos^2 x)^2 \sin x \, dx = - \int (1 - u^2)^2 \, du = - \int (1 - 2u^2 + u^4) \, du$$

$u = \cos x$
 $du = -\sin x$

$$- \left(\cos x - 2 \frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} \right) + C$$

$$\frac{d}{dx} \left(\cos x - \frac{2}{3} \cos^3 x + \frac{1}{5} \cos^5 x \right) = -\sin x + 2 \cos^2 x \sin x - \cos^4 x \sin x$$

$$= \sin x (1 - 2 \cos^2 x + \cos^4 x)$$

$$= \sin x (1 - \cos^2 x)^2 \checkmark$$

2. (8 points) Find $\int \sin^4 x \, dx = \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$= \frac{1}{4} \int (1 - 2 \cos 2x + \frac{1}{2}(1 + \cos 4x)) \, dx$$

$$= \frac{1}{4} \left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8} \right) + C$$

3. (8 points) Where does the function $f(x) = \sum_{n=1}^{\infty} \frac{(x-7)^n}{n \cdot 3^n}$ converge? Justify your answer.

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x-7|^{n+1}}{n+1 \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{|x-7|^n}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{n+1} \frac{|x-7|}{3} = \frac{|x-7|}{3}$$

If $\rho < 1$, the $f(x)$ converges
If $\rho > 1$, then $f(x)$ diverges

$$\rho < 1 \text{ when } \frac{|x-7|}{3} < 1$$

$$|x-7| < 3$$

$$-3 < x-7 < 3$$

$$4 < x < 10$$

$$\text{at } 4 \quad f(4) = \sum \frac{(-3)^n}{n 3^n} = \sum \frac{(-1)^n}{n}$$

This is the alternating Harmonic series. It converges

$$\text{at } 10 \quad f(10) = \sum \frac{3^n}{n 3^n} = \sum \frac{1}{n}$$

This is the Harmonic series. It diverges

$f(x)$ converges on

$$4 \leq x < 10$$

$f(x)$ diverges elsewhere else.

4. (8 points) Solve the differential equation $\frac{dy}{dt} = 6y$ with the initial condition $y(1) = 4$. Check your answer.

$$\int \frac{dy}{y} = \int 6 dt$$

$$\ln y = 6t + c$$

$$e^{\ln y} = e^{6t} e^c$$

$$y = k e^{6t}$$

$$4 = k e^6$$

$$\frac{4}{e^6} = k$$

$$y = \frac{4}{e^6} e^{6t}$$

$$\text{check } y(1) = \frac{4}{e^6} e^6 = 4 \checkmark$$

$$\frac{dy}{dt} = 6 \frac{4}{e^6} e^{6t} = 6y \checkmark$$

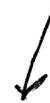
5. (8 points) Does $\sum_{n=1}^{\infty} 1 - \frac{2}{n}^n$ converge? Justify your answer.

The limit of the n^{th} term is $\lim_{n \rightarrow \infty} \left(1 - \frac{2}{n}\right)^n = e^{-2}$

This limit is not zero. Thus $\sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^n$ diverges by the n^{th} term test.

6. (8 points) Does $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ converge? Justify your answer.

The series
"telescopes"



The m^{th} partial sum is $\sum_{h=1}^m \frac{1}{h} - \frac{1}{h+1} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{m} - \frac{1}{m+1}\right)$

$$1 - \frac{1}{m+1}$$

The sum of a series is the limit of the sequence of partial sums. The sum $\sum_{h=1}^{\infty} \frac{1}{h} - \frac{1}{h+1} = \lim_{m \rightarrow \infty} \text{sum} = \lim_{m \rightarrow \infty} \left(1 - \frac{1}{m+1}\right) = 1$

Thus $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ converges to 1.

7. (8 points) Does $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge? Justify your answer.

$$\text{test } \rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{1}{2} = \frac{1}{2} <$$

$\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by the ratio

8. (8 points) Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^5}$ converge? Justify your answer.

Compare to $\sum \frac{1}{n^4}$ which is the p-series with $p=4 > 1$ so $\sum \frac{1}{n^4}$

converges. $\ln n < n$ so $\frac{\ln n}{n^5} < \frac{n}{n^5} = \frac{1}{n^4}$

$\therefore \sum_{n=1}^{\infty} \frac{\ln n}{n^5}$ also converges

9. (8 points) Suppose that the government pumps an extra \$1 billion into the economy. Assume that each business and individual saves 25% of its income and spends the rest, so that of the initial \$1 billion, 75% is respent by individuals and businesses. Of that amount, 75% is spent, and so forth. What is the total increase in spending due to the government action?

$$\text{Extra spending} = B + \frac{3}{4}B + \left(\frac{3}{4}\right)^2 B + \left(\frac{3}{4}\right)^3 B + \dots = \frac{B}{1 - \frac{3}{4}} = \text{\$4 billion}$$

where $B = \text{\$1 Billion}$

Geometric series

ratio $\frac{3}{4}$

$a = B$

10. (8 points) Find the general solution of $\frac{dy}{dx} - \frac{y}{x} = 3x^3$. Check your answer.

This problem has the form $y' + P(x)y = Q(x)$

Multiply both sides by $\mu = e^{\int P(x) dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

$$\frac{1}{x} y' - \frac{y}{x^2} = 3x^2$$

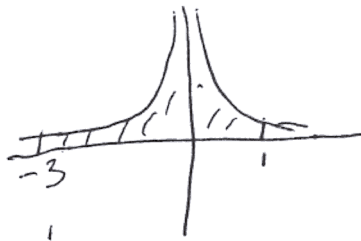
The left side is $\frac{d}{dx} \left(\frac{1}{x} y \right)$

$$\text{So } \frac{1}{x} y = \int 3x^2 dx = x^3 + C$$

$$y = x^4 + Cx$$

$$\frac{dy}{dx} - \frac{y}{x} = 4x^3 + C - \frac{x^4 + Cx}{x} = 4x^3 - 3x^3 = x^3$$

11. (7 points) Find $\int_{-3}^1 \frac{1}{x^2} dx$.



$$\text{integral} = \lim_{a \rightarrow 0^-} \int_{-3}^a \frac{1}{x^2} dx + \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{x^2} dx$$

$$= \lim_{a \rightarrow 0^-} \left[-\frac{1}{x} \right]_{-3}^a + \lim_{b \rightarrow 0^+} \left[-\frac{1}{x} \right]_b^1$$

$$= \lim_{a \rightarrow 0^-} \left[-\frac{1}{a} - \frac{1}{-3} \right] + \lim_{b \rightarrow 0^+} \left[-\frac{1}{1} + \frac{1}{b} \right]$$

$$= +\infty - \frac{1}{3} - 1 + \infty$$

The integral diverges to $+\infty$.

12. (7 points) Find $\int \sqrt{1+x^2} dx$. (Be sure to check your answer.)

$$x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

$$1+x^2 = 1 + \tan^2 \theta = \sec^2 \theta$$

$$\text{Integral} = \int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta$$

$$u = \sec \theta \quad v = \tan \theta$$

$$du = \sec \theta \tan \theta d\theta \quad dv = \sec^2 \theta d\theta$$

$$\text{so } \int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta$$

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta$$

$$\int \sec^3 \theta d\theta = \frac{1}{2} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right] + C$$

$$\checkmark: \frac{d}{dx} \text{IA}$$

$$= \frac{1}{2} \left[\frac{x^2}{\sqrt{1+x^2}} + \sqrt{1+x^2} + \frac{\frac{x}{\sqrt{1+x^2}} + 1}{\sqrt{1+x^2} + x} \right]$$

$$= \frac{1}{2} \left[\frac{x^2 + 1 + x^2}{\sqrt{1+x^2}} + \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}(\sqrt{1+x^2} + x)} \right]$$

$$= \frac{1}{2} \left[\frac{2x^2 + 1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right]$$

$$= \frac{x^2 + 1}{\sqrt{1+x^2}} = \sqrt{1+x^2} \checkmark$$

$$\text{ans} = \frac{1}{2} \left[\sqrt{1+x^2} x + \ln |\sqrt{1+x^2} + x| \right] + C$$

13. (7 points) Find $\int \frac{x}{\sqrt{9-16x^2}} dx$ (Be sure to check your answer.)

$$u = 9 - 16x^2$$

$$du = -32x dx$$

$$\text{integral} = \frac{-1}{32} \int u^{-\frac{1}{2}} du = \frac{-1}{32} 2 u^{\frac{1}{2}} + C = \frac{-1}{16} \sqrt{9-16x^2} + C$$

$$\frac{d}{dx} (PA) = \frac{-1}{16} \frac{(-16x) \cdot 2}{2\sqrt{9-16x^2}} \quad \checkmark$$

14. (7 points) Find $\int \frac{1}{\sqrt{9-16x^2}} dx$. (Be sure to check your answer.)

$$u = 4x$$

$$du = 4dx$$

$$\text{integral} = \frac{1}{4} \int \frac{du}{\sqrt{9-u^2}} = \frac{1}{4} \sin^{-1}\left(\frac{4x}{3}\right) + C$$

$$\int \frac{du}{\sqrt{9-u^2}} = \sin^{-1} \frac{u}{3} + C$$

$$\checkmark: \frac{d}{dx} (PA) = \frac{1}{4} \frac{\frac{4}{3}}{\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \frac{1}{\sqrt{9-\frac{16x^2}{9}}} \quad \checkmark$$

15. (7 points) Find $\int \ln x \, dx$. (Be sure to check your answer.)

$$u = \ln x \quad v = x$$

$$dv = \frac{1}{x} dx \quad du = dx$$

$$x \ln x - \int dx = \boxed{x \ln x - x + C}$$

$$\checkmark: \frac{d}{dx} (x \ln x - x + C) = x \frac{1}{x} + \ln x - 1 = \ln x \checkmark$$

16. (7 points) Graph $r = 2 - 4 \sin \theta$. Find the area inside the inner loop of this graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
r	2	0	-2	0	2	6	2



$$\frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2 - 4 \sin \theta)^2 d\theta = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 4 - 16 \sin \theta + \frac{16}{2} (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{2} \left(12\theta + 16 \cos \theta - 8 \frac{\sin 2\theta}{2} \right) \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{1}{2} \left(12 \left(\frac{4\pi}{6} \right) - 32 \frac{\sqrt{3}}{2} + 8 \frac{\sqrt{3}}{2} \right)$$

$$\boxed{\frac{8\pi - 12\sqrt{3}}{2}}$$

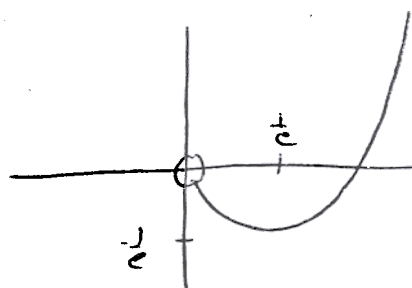
17. (7 points) Let $f(x) = x \ln x$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of $y = f(x)$. Graph $y = f(x)$.

$$f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

$f' = 0$ at $x = \frac{1}{e}$

$$f'' = \frac{1}{x}$$

f'' always positive
Note the domain: $0 < x$



$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$$

$x \rightarrow 0$
 f is inc for $x > \frac{1}{e}$

f is dec for $0 < x < \frac{1}{e}$

f is always concave up

$(\frac{1}{e}, -\frac{1}{e})$ is the local min

no local max
no point of inf.

18. (7 points) Find $\int \frac{3 + 2x + 7x^2 - 3x^3}{x^4 + x^2} dx$. (Be sure to check your answer.)

$$\frac{3 + 2x + 7x^2 - 3x^3}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

$$\text{ans} = 2 \ln|x| + \frac{3}{x} - \frac{5}{2} \ln|x^2 + 1| + 4 \tan^{-1} x + C$$

$$3 + 2x + 7x^2 - 3x^3 = A(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2$$

$$= Ax^3 + Bx^2 + Ax + B + Cx^3 + Dx^2$$

$$3 = B$$

$$A = 2$$

$$7 = B + D \text{ so } D = 4$$

$$-3 = A + C \text{ so } C = -5$$

check $\frac{2}{x} + \frac{3}{x^2} + \frac{-5x + 4}{x^2 + 1}$

$$= \frac{2x(x^2 + 1) + 3(x^2 + 1) + (-5x + 4)x^2}{x^2(x^2 + 1)}$$

$$\frac{-3x^3 + 7x^2 + 2x + 3}{x^2(x^2 + 1)}$$

19. (7 points) Which familiar function is equal to $\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \frac{x^4}{6!} + \frac{x^5}{7!} + \dots$
Justify your answer.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{x-1} = \frac{x^0}{1!} + \frac{x^1}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots$$

$$\frac{e^{x-1}}{x^2} = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \dots$$

20. (7 points) Find the third Taylor polynomial $P_3(x)$ for $f(x) = \ln x$ about $a = 1$. Estimate the error that is introduced if $f(x)$ is approximated by $P_3(x)$ for $.8 \leq x \leq 1.2$.

$$f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4}$$

$$P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$$|f(x) - P_3(x)| = |R_3(x)| = \left| \frac{f^{(4)}(c)}{4!} (x-1)^4 \right| = \left| \frac{6}{24 c^4} \right| |x-1|^4$$

for some c between x and 1 .

$$\leq \frac{1}{4(0.8)^4} (0.2)^4$$

$\frac{1}{c^4}$ is largest when c is smallest
 $.8 \leq x \leq 1.2 \Rightarrow -0.2 \leq x-1 \leq 0.2$
 so $|x-1| \leq 0.2$

$$\text{So } |f(x) - P_3(x)| \leq \frac{1}{4(4)^4}$$