Math 142, Fall 2000, Exam 3, Solution to Problems 3, 4, 5, 6, 7, and 8.

3. Find $\int \sqrt{1-x^2}\,dx$.
Let $x=\sin\theta$ **. It follows that** $dx=\cos\theta d\theta$ **and**

$$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta.$$

We see that

$$\int \sqrt{1-x^2}\,dx=\int \cos^2\theta\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)\,d\theta=\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C.$$

Recall that

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

We conclude that

$$\int \sqrt{1-x^2}\,dx=\boxed{\frac{1}{2}\left(\arcsin\theta+x\sqrt{1-x^2}\right)+C.}$$

4. Find $\int xe^x\,dx$.
Use integration by parts with

$$\begin{array}{ll} u=x & v=e^x \\ du=dx & dv=e^x\,dx. \end{array}$$

We see that

$$\int xe^x\,dx = xe^x-\int e^x\,dx=\boxed{xe^x-e^x+C.}$$

The derivative of the proposed answer is equal to $xe^x+e^x-e^x$ **.** ✓

5. Find $\int xe^{x^2}\,dx$.
Let $u=x^2$ **. It follows that** $du=2xdx$ **and**

$$\int xe^{x^2}\,dx=\frac{1}{2}\int e^u\,du=\frac{1}{2}e^u+C=\boxed{\frac{1}{2}e^{x^2}+C}.$$

Notice that the derivative of $\frac{1}{2}e^{x^2}$ **is** xe^{x^2} **, as expected.**

6. Find $\int_{-3}^{1}\frac{1}{x^2}\,dx$.
The graph of $y=\frac{1}{x^2}$ **is always above the** x **-axis. The answer, represents an area, and therefore, must be positive. The function** $f(x)=\frac{1}{x^2}$ **goes to**

plus ∞ as x goes to zero. We must treat this integral as an improper integral:

$$\int_{-3}^{1} \frac{1}{x^2}\,dx = \lim_{b\to 0^-} -\frac{1}{x}\Big|_{-3}^{b} + \lim_{a\to 0^+} -\frac{1}{x}\Big|_{a}^{1} = \lim_{b\to 0^-} -\frac{1}{b} + \frac{1}{3} + \lim_{a\to 0^+} -1 + \frac{1}{a}$$

$$= +\infty + \frac{1}{3} - 1 + \infty = \boxed{+\infty.}$$

This integral is most certainly NOT equal to $+\frac{1}{3} - 1 = -\frac{2}{3}$.

7. Find $\lim\limits_{x\to\infty} \dfrac{e^{-x}}{x^{-1}}$.

We see that $\lim\limits_{x\to\infty} \dfrac{e^{-x}}{x^{-1}} = \lim\limits_{x\to\infty} \dfrac{x}{e^x}$. L'hopital's rule applies because the top and the bottom are both going to infinity. So,

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{-1}} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \boxed{0}.$$

8. Find $\displaystyle\int \frac{2x^3+5x^2+x+3}{x^2(x^2+1)}\,dx$.

Write

$$\frac{2x^3+5x^2+x+3}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}.$$

Multiply both sides by $x^2(x^2+1)$ to get

$$2x^3+5x^2+x+3 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2;$$

hence

$$2x^3+5x^2+x+3 = (A+C)x^3 + (B+D)x^2 + Ax + B.$$

Equate the corresponding coefficients to see that

$$\begin{aligned} 2 &= A + C \\ 5 &= B + D \\ 1 &= A \\ 3 &= B. \end{aligned}$$

We see that $C = 1$ and $D = 2$. Thus,

$$\int \frac{2x^3+5x^2+x+3}{x^2(x^2+1)}\,dx = \int \frac{1}{x} + \frac{3}{x^2} + \frac{x+2}{x^2+1}\,dx$$

$$= \boxed{\ln|x| - \frac{3}{x} + \frac{1}{2}\ln(x^2+1) + 2\arctan x + C.}$$

We check that the derivative of the proposed answer is

$$\frac{1}{x} + \frac{3}{x^2} + \frac{x}{x^2+1} + \frac{2}{1+x^2} = \frac{x(x^2+1) + 3(x^2+1) + x^3 + 2x^2}{x^2(1+x^2)}$$

$$= \frac{2x^3+5x^2+x+3}{2x^2(1+x^2)}.\checkmark$$