Math 142, Exam 3, Solutions Fall 2010

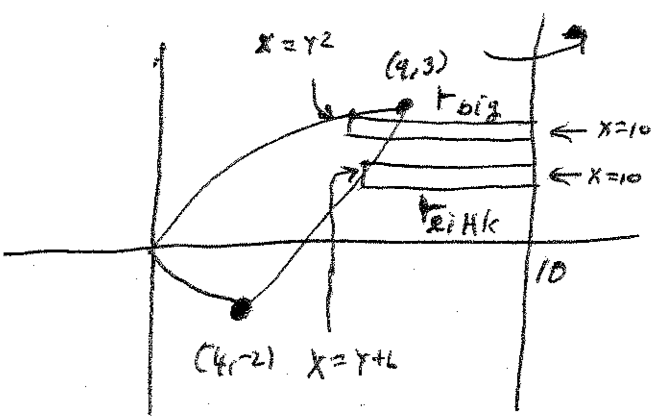
Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

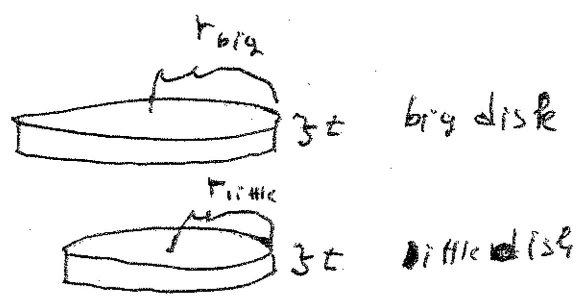
The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (6 points) **Consider the region bounded by** $x = y^2$ **and** $y = x - 6$. **Revolve the region about** $x = 10$. **Find the volume of the resulting solid.**

We find the intersection points by solving $y = y^2 - 6$. This is $0 = y^2 - y - 6$, or $0 = (y-3)(y+2)$. So, $y = -2$ or $y = 3$. The intersection points are $(9,3)$ and $(4,-2)$. We draw the parabola and the line. The line $x = 10$ is a vertical line to the right of the region. The easiest way to do the problem is by using big disks minus little disks. We partition the y-axis from $y = -2$ to $y = 3$ and we express everything in terms of y. In particular the thickness of each disk is $t = dy$.





The radius of the big disc is $10 - y^2$. The radius of the little disc is $10 - (y+6)$. The volume of the big disk is $\pi r^2 t = \pi(10 - y^2)^2 dy$. The volume of the small disk is $\pi r^2 t = \pi(4-y)^2 dy$. The volume of the solid is

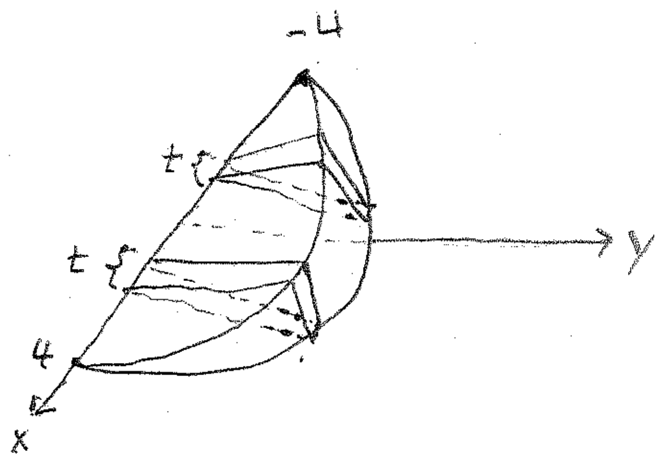
$$\pi \int_{-2}^{3} [(10-y^2)^2 - (4-y)^2] dy = \pi \int_{-2}^{3} [100 - 20y^2 + y^4 - (16 - 8y + y^2)] dy$$

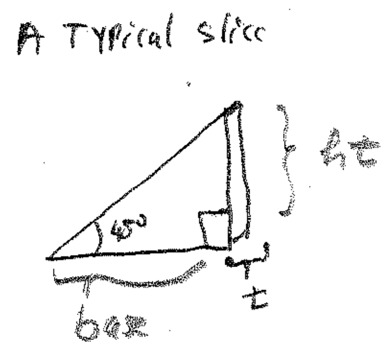
$$= \pi \int_{-2}^{3} [84 + 8y - 21y^2 + y^4] dy = \pi \left(\left(84y + 4y^2 - \frac{21}{3}y^3 + \frac{y^5}{5} \right) \Big|_{-2}^{3} \right)$$

$$= \boxed{\pi\left(84(3)+4(9)-\frac{21}{3}27+\frac{3^5}{5}-\left(84(-2)+4(4)-\frac{21}{3}(-2)^3+\frac{(-2)^5}{5}\right)\right)}.$$

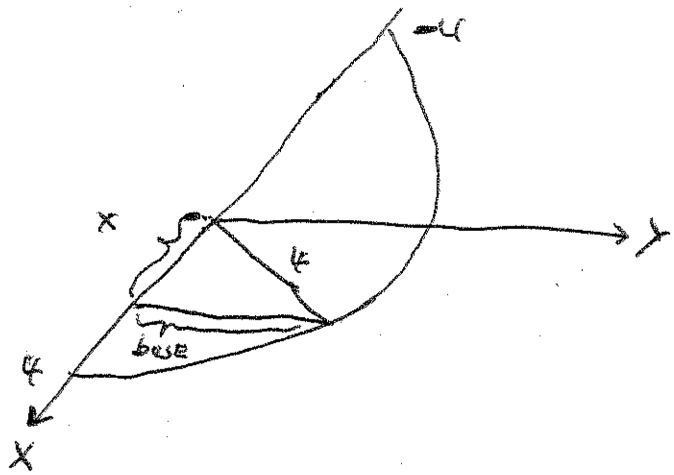
2. (6 points) **Consider a wedge cut from a cylinder of radius 4. This wedge is cut using 2 planes. The first plane is perpendicular to the axis of the cylinder. The second plane intersects the first plane through a diameter of the cylinder. The angle of intersection of the two planes is 45 degrees. Find the volume of the wedge.**

We view the first plane as the xy-plane. We make the intersection of the two planes occur on the x-axis from $x=-4$ to $x=4$. We slice the wedge into a collection of triangular slices by partitioning the x-axis from $x=-4$ to $x=4$. Each slice has thickness $t=dx$. Each slice has a base and a height. The volume of a given slice is $\frac{1}{2}$ base times height times thickness. The base and the height are equal (because the angle of intersection of the two planes is 45 degrees).





The base depends on which slice we are studying. In other words, the base depends on the x-coordinate of our slice. Indeed the base is the y-coordinate on the circle with center $(0,0)$ and radius 4.



So the base is $\sqrt{16-x^2}$. The volume of the slice with x-coordinate x is $(1/2)$ base times height times thickness, which equals $(1/2)\sqrt{16-x^2}\sqrt{16-x^2}dx = (1/2)(16-x^2)dx$. The volume of the solid is

$$\tfrac{1}{2}\int_{-4}^{4}(16-x^2)dx = \tfrac{1}{2}(16(x) - \tfrac{x^3}{3})\Big|_{-4}^{4}$$

$$= 2\tfrac{1}{2}(64 - \tfrac{64}{3}) = \tfrac{2}{3}(64) = \boxed{\frac{128}{3}}.$$

3. (6 points) **Consider the sequence $\{a_n\}$ with $a_0 = 1$, and for all $n \geq 1$, $a_n = \sqrt{2a_{n-1}}$. Prove that this sequence is increasing. Prove that this sequence is bounded. Deduce that the sequence converges. Find the limit of the sequence.**

We see that $1 < \sqrt{2} < \sqrt{2\sqrt{2}}$. So the series starts out to be increasing. Assume that after a while we have $a_{n-1} < a_n$. Then $2a_{n-1} < 2a_n$ and $\sqrt{2a_{n-1}} < \sqrt{2a_n}$. But $\sqrt{2a_{n-1}} = a_n$ and $\sqrt{2a_n} = a_{n+1}$. In other words, we have shown that: if a_{n-1} is less than a_n, then a_n is also less than a_{n+1}. So this sequence keeps on increasing forever. (This technique is called Mathematical Induction!)

We also use Mathematical Induction to show that 2 is an upper bound for this sequence. We have $1 < 2$, and $\sqrt{2} < 2$. Suppose $a_{n-1} < 2$, then $2a_{n-1} < 2 \cdot 2$ and $\sqrt{2a_{n-1}} < \sqrt{2 \cdot 2}$. But $\sqrt{2a_{n-1}}$ is also called a_n and $\sqrt{2\cdot 2}$ is also called 2. We have shown that if a_{n-1} is less than 2, then a_n is also less than 2. So this sequence keeps on being less than 2 forever!

Our sequence is an increasing bounded sequence. The Completeness Axiom guarantees that our sequence has a limit. Let $L = \lim\limits_{n\to\infty} a_n$. We may take $\lim\limits_{n\to\infty}$ of both sides of $a_n = \sqrt{2a_{n-1}}$ to see that $\lim\limits_{n\to\infty} a_n = \sqrt{2 \lim\limits_{n\to\infty} a_{n-1}}$; so, $L = \sqrt{2L}$ or $L^2 = 2L$. Thus, $L = 0$ or $L = 2$. (But L can not be 0 because every element of the sequence is at least 1.) Thus, $L = 2$.

4. (6 points) **Express the repeating decimal $r = 1.53\overline{42}$ as the ratio of two integers. Explain what you are doing very thoroughly.**

We see that $100r - r$ is

$$\begin{array}{r} 153.42\overline{42} \\ -1.53\overline{42} \end{array}$$

So, $99r = 151.89$. In other words, $r = \frac{151.89}{99} = \boxed{\dfrac{15189}{9900}}.$

Also, r is 1.53 plus the sum of the geometric series with initial term $.0042$ and ratio $1/100$. In other words,

$$r = 1.53 + \frac{.0042}{1-\frac{1}{100}} = 1.53 + \frac{.0042}{\frac{99}{100}} = 1.53 + \frac{(.0042)100}{99} = \frac{151.47 + (.42)}{99} = \frac{151.89}{99}$$

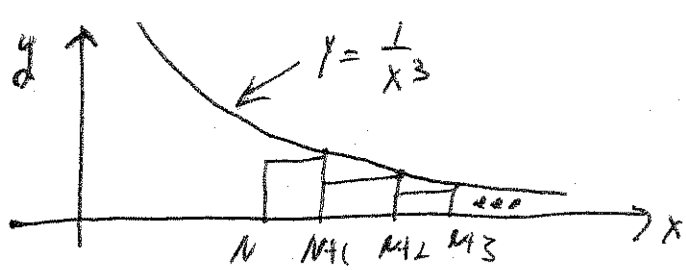
$$= \frac{15189}{9900}.$$

5. (6 points) **Approximate $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ with an error of at most $\frac{1}{100}$. Explain what you are doing very thoroughly.**

It is clear that

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^3} \right| = \sum_{n=N+1}^{\infty} \frac{1}{n^3}.$$

We look at the picture



to see that $\sum\limits_{n=N+1}^{\infty} \frac{1}{n^3}$, which is the area inside the boxes, is less than the area under the curve, which is

$$\int_N^\infty \frac{1}{x^3}dx = \lim_{b\to\infty} \frac{-1}{2x^2}\bigg|_N^b = \lim_{b\to\infty}\left(\frac{-1}{2b^2}+\frac{1}{2N^2}\right) = \frac{1}{2N^2}.$$

We have shown that

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{N}\frac{1}{n^3}\right| \le \frac{1}{2N^2}.$$

We want to make $\frac{1}{2N^2} \le \frac{1}{100}$. So we want $50 < N^2$. Take $8 = N$. We have shown that

$$\boxed{\sum_{n=1}^{8}\frac{1}{n^3} \text{ approximates } \sum_{n=1}^{\infty}\frac{1}{n^3} \text{ with an error of at most } \frac{1}{100}.}$$

6. (6 points) **Approximate** $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$ **with an error of at most** $\frac{1}{100}$. **Explain what you are doing very thoroughly.**

We use the Alternating Series Test. We see that the series is an alternating series. We see that $\frac{1}{1^3} > \frac{1}{2^3} > \frac{1}{3^3}\cdots$. We see that $\lim\limits_{n\to\infty}\frac{1}{n^3} = 0$. The Alternating Series Test tells us that

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3} - \sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^3}\right| \le \frac{1}{(N+1)^3}.$$

We want $\frac{1}{(N+1)^3} \le \frac{1}{100}$. We want $100 < (N+1)^3$. We see that

$$100 < 125 = 5^3 = (4+1)^3.$$

Take $N = 4$. We have shown that

$$\boxed{\sum_{n=1}^{4}\frac{(-1)^{n+1}}{n^3} \text{ approximates } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3} \text{ with an error of at most } \frac{1}{100}.}$$

7. (7 points) **Does** $\sum\limits_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ **converge? Justify your answer very thoroughly.**

Use the Integral Test. We see that $f(x) = \frac{1}{x\sqrt{\ln x}}$ is a positive decreasing function. The Integral Test tells us that $\sum\limits_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ and $\int_2^\infty \frac{1}{x\sqrt{\ln x}} dx$ both converge or both diverge. We compute

$$\int_2^\infty \frac{1}{x\sqrt{\ln x}} dx = \lim_{b\to\infty} 2\sqrt{\ln x}\Big|_2^b = \lim_{b\to\infty} (2\sqrt{\ln b} - 2\sqrt{\ln 2}) = \infty.$$

The integral diverges. So, $\boxed{\text{the series } \sum\limits_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \text{ also diverges.}}$

8. (7 points) **Does** $\sum\limits_{n=1}^{\infty} n(\frac{2}{3})^n$ **converge? Justify your answer very thoroughly.**

Use the Ratio Test. We compute that

$$\rho = \lim_{n\to\infty} \frac{(n+1)(\frac{2}{3})^{n+1}}{n(\frac{2}{3})^n} = \lim_{n\to\infty} \frac{n+1}{n}\frac{2}{3} = \lim_{n\to\infty} (1 + \frac{1}{n})\frac{2}{3} = \frac{2}{3}.$$

We see that $\rho < 1$. We conclude that $\boxed{\sum\limits_{n=1}^{\infty} n(\frac{2}{3})^n \text{ converges.}}$