

Math 142, Exam 2, Fall 2009 Solutions

PRINT Your Name: \_\_\_\_\_

When does your FRIDAY class meet? \_\_\_\_\_

Write your answers as legibly as you can.

There are 10 problems on 5 pages. The exam is worth 100 points. Each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. **CIRCLE** your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website a few hours after the exam is finished.

1. Find  $\int \frac{e^x dx}{1 + e^x}$ . Check your answer.

Let  $u = e^x$ . It follows that  $du = e^x dx$ . The integral is

$$\int \frac{du}{u} = \ln |u| + C = \boxed{\ln(1 + e^x) + C.}$$

**Check:** The derivative of the proposed answer is  $\frac{e^x}{1+e^x} \cdot \checkmark$

2. Compute  $\int_{-1}^3 \frac{1}{x^2} dx$

The function  $f(x) = \frac{1}{x^2}$  goes to infinity as  $x$  approaches zero. For all  $x$  other than zero,  $f(x)$  is positive. Thus, this integral represents an area. Either this integral is infinite; or else, the integral is finite and positive.

$$\begin{aligned} \int_{-1}^3 \frac{1}{x^2} dx &= \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{x^2} dx + \lim_{a \rightarrow 0^+} \int_a^3 \frac{1}{x^2} dx \\ &= \lim_{b \rightarrow 0^-} \left. \frac{-1}{x} \right|_{-1}^b + \lim_{a \rightarrow 0^+} \left. \frac{-1}{x} \right|_a^3 = \lim_{b \rightarrow 0^-} \left( \frac{-1}{b} - \frac{-1}{-1} \right) + \lim_{a \rightarrow 0^+} \left( \frac{-1}{3} - \frac{-1}{a} \right). \end{aligned}$$

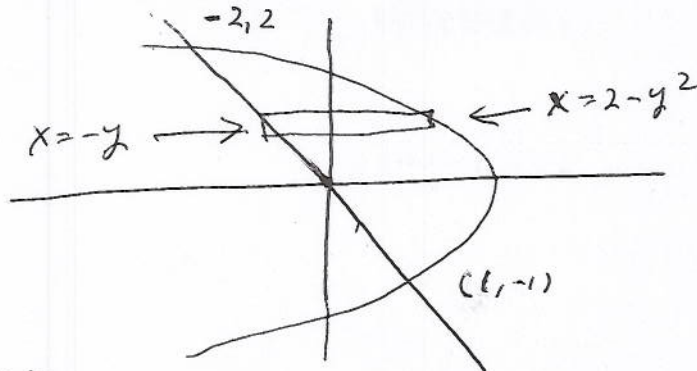
Notice that

$$\lim_{b \rightarrow 0^-} \left( \frac{-1}{b} \right) = +\infty \quad \text{and} \quad \lim_{a \rightarrow 0^+} \left( -\frac{1}{a} \right) = +\infty.$$

Thus, the integral diverges to  $+\infty$ . Notice that  $-\frac{4}{3}$  has NOTHING TO DO WITH THE FINAL ANSWER. An "answer"  $-\frac{4}{3}$  will receive a score of 0.

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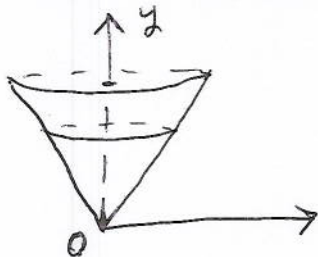
3. Find the area between  $x + y = 0$  and  $2 = x + y^2$ .



The area is

$$\int_{-1}^2 2 - y^2 - (-y) dy = 2y - \frac{y^3}{3} + \frac{y^2}{2} \Big|_{-1}^2 = \boxed{4 - \frac{8}{3} + 2 - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)}$$

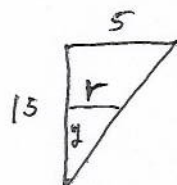
4. Suppose that a conical tank is filled with oil which has a density of 50 lb/ft<sup>3</sup>. The radius at the top of the tank is 5 ft and the tank is 15 ft high. How much work is done in pumping the oil over the edge of the tank? Be sure to include units in your answer.



Be sure to draw and label an axis. We have chosen the  $y$ -axis to point upward with  $y = 0$  being the bottom of the tank. The work to lift the layer of oil with  $y$ -coordinate  $y$  is

Force · distance = weight · distance = volume · density · distance =  $\pi r^2 t \cdot 50 \cdot$  distance,

where  $t = dy$ ,  $r$  is computed from similar triangles  $\frac{r}{y} = \frac{5}{15}$ :



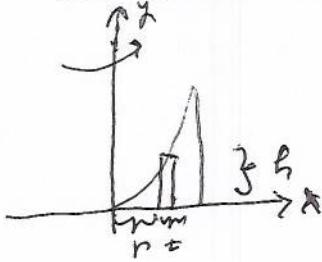
So,  $r = \frac{1}{3}y$ . The distance to lift the layer of oil whose  $y$ -coordinate is  $y$  is  $15 - y$ . The work to lift one layer of oil is:

$$\pi\left(\frac{1}{3}y\right)^2 50(15 - y)dy.$$

The work to lift all of the oil is

$$\begin{aligned} \frac{50\pi}{9} \int_0^{15} (15y^2 - y^3)dy &= \frac{50\pi}{9} \left( \frac{15y^3}{3} - \frac{y^4}{4} \right) \Big|_0^{15} = \frac{50(15)^4\pi}{9} \left( \frac{1}{3} - \frac{1}{4} \right) \\ &= \boxed{\frac{50(15)^4\pi}{9 \cdot 12} \text{ ft-lb.}} \end{aligned}$$

5. Consider the region bounded by  $y = x^3$ ,  $x = 1$ , and  $y = 0$ . Revolve this region about the  $y$ -axis. Find the volume of the resulting solid.



Spin the rectangle. Get a shell of volume  $2\pi rht$ , where  $t = dx$ ,  $r = x$ , and  $h = x^3$ . The volume of the solid is

$$2\pi \int_0^1 x^4 dx = \frac{2\pi x^5}{5} \Big|_0^1 = \boxed{\frac{2\pi}{5}}$$

6. Find the limit of the sequence whose  $n^{\text{th}}$  term is  $a_n = \left(\frac{3+n}{n}\right)^n$ .

The answer is

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{n+3}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right)^n = \boxed{e^3}$$

because we know that  $\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r$ .

7. Find  $\int \frac{3x-4}{x^2-3x+2} dx$ . Check your answer.

Multiply both sides of

$$\frac{3x-4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

by  $(x-1)(x-2)$  to get

$$3x-4 = A(x-2) + B(x-1).$$

Plug in  $x=1$  to learn  $A=1$ . Plug in  $x=2$  to learn  $B=2$ . Check that

$$\frac{1}{x-1} + \frac{2}{x-2} = \frac{x-2+2x-2}{x^2-3x+2} = \frac{3x-4}{x^2-3x+2} \checkmark.$$

Now we do the integral. The original problem is equal to

$$\int \frac{1}{x-1} + \frac{2}{x-2} dx = \boxed{\ln|x-1| + 2\ln|x-2| + C.}$$

8. Find  $\int \frac{3x^2+8x+10}{x^3+2x^2+5x} dx$ . Check your answer.

Multiply both sides of

$$\frac{3x^2+8x+10}{x(x^2+2x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5}$$

by  $x(x^2+2x+5)$  to see that:

$$3x^2+8x+10 = A(x^2+2x+5) + (Bx+C)x$$

$$3x^2+8x+10 = (A+B)x^2 + (2A+C)x + 5A$$

Equate the corresponding coefficients:

$$3 = A+B, \quad 8 = 2A+C, \quad 10 = 5A;$$

hence,  $A=2$ ,  $C=4$ , and  $B=1$ . Check that

$$\frac{2}{x} + \frac{x+4}{x^2+2x+5} = \frac{2x^2+4x+10+x^2+4x}{x^3+2x^2+5x} = \frac{3x^2+8x+10}{x^3+2x^2+5x}.$$

Before we integrate, we see that

$$x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4$$

The problem is

$$\begin{aligned} \int \frac{2}{x} + \frac{x+4}{x^2+2x+5} dx &= \int \frac{2}{x} + \frac{x+1}{(x+1)^2+4} + \frac{3}{(x+1)^2+4} dx \\ &= \boxed{2\ln|x| + \frac{1}{2}\ln((x+1)^2+4) + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C} \end{aligned}$$

9. Find  $\int \ln x dx$ . Check your answer.

Use integration by parts. Let  $u = \ln x$  and  $dv = dx$ . It follows that  $du = \frac{dx}{x}$  and  $v = x$ . The original problem is  $\int u dv = uv - \int v du$

$$= x \ln x - \int dx = \boxed{x \ln x - x + C.}$$

**Check:** The derivative of the proposed answer is

$$x \frac{1}{x} + \ln x - 1 = \ln x \checkmark.$$

10. Find  $\int \frac{dx}{(1-x^2)^{3/2}}$ . Check your answer.

Let  $x = \sin \theta$ . It follows that  $dx = \cos \theta d\theta$ . It also follows that

$$(1-x^2) = 1 - \sin^2 \theta = \cos^2 \theta.$$

The integral is equal to

$$\int \frac{\cos \theta d\theta}{\cos^3 \theta} = \int \frac{d\theta}{\cos^2 \theta} = \int \sec^2 \theta d\theta = \tan \theta + C = \frac{\sin \theta}{\cos \theta} + C = \boxed{\frac{x}{\sqrt{1-x^2}} + C.}$$

**Check:** The derivative of the proposed answer is

$$\begin{aligned} x(-1/2)(1-x^2)^{-3/2}(-2x) + \frac{1}{\sqrt{1-x^2}} \\ = \frac{x^2 + 1 - x^2}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{3/2}} \checkmark \end{aligned}$$