

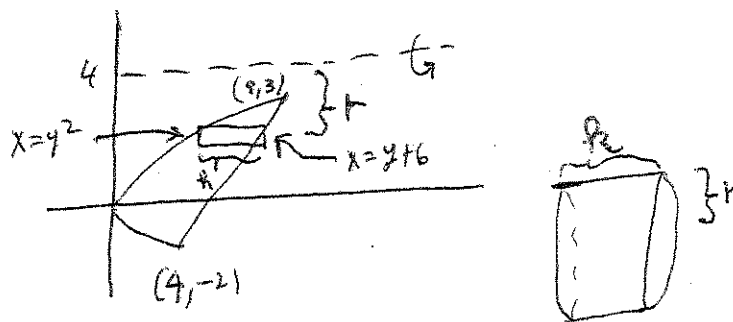
Math 142, Exam 4, Solutions Fall 2010

Write everything on the blank paper provided. You should **KEEP** this piece of paper. If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it - I will still grade your exam.

The exam is worth 50 points. SHOW your work. CIRCLE your answer.  
**No Calculators or Cell phones.**

1. (6 points) Consider the region bounded by  $x = y^2$  and  $y = x - 6$ . Revolve the region about  $y = 4$ . Find the volume of the resulting solid.

We find the intersection points by solving  $y = y^2 - 6$ . This is  $0 = y^2 - y - 6$ , or  $0 = (y - 3)(y + 2)$ . So,  $y = -2$  or  $y = 3$ . The intersection points are  $(9, 3)$  and  $(4, -2)$ . We draw the parabola and the line. The line  $y = 4$  is a horizontal line above the region. The easiest way to do the problem is by using shells. We partition the  $y$ -axis from  $y = -2$  to  $y = 3$  and we express everything in terms of  $y$ . In particular the thickness of each shell is  $t = dy$ .

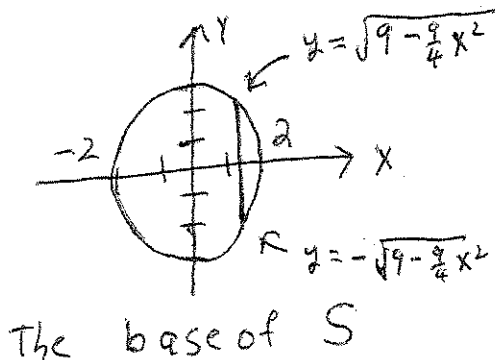


The radius of the shell with  $y$ -coordinate  $y$  is the big  $y$ -coordinate (namely 4) minus the little  $y$ -coordinate (namely  $y$ ); so,  $4 - y$ . The height of the shell is the big  $x$ -coordinate minus the little  $x$ -coordinate; so,  $(y + 6) - y^2$ . The volume of the shell with  $y$ -coordinate  $y$  is  $2\pi rht = 2\pi(4 - y)(y + 6 - y^2)dy$ . The volume of the solid is

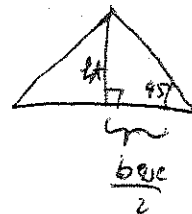
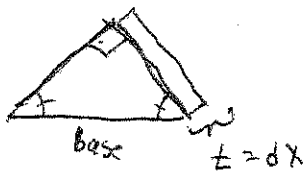
$$2\pi \int_{-2}^3 (4 - y)(y + 6 - y^2)dy.$$

2. (6 points) Consider a solid  $S$ . The base of  $S$  is an elliptical region with boundary curve  $9x^2 + 4y^2 = 36$ . The cross sections of  $S$  perpendicular to the  $x$ -axis are isosceles right triangles with hypotenuse in the base. Find the volume of  $S$ .

The boundary of the base is an ellipse. An ellipse is a slightly eccentric circle. We see that when  $x = 0$ , then the ellipse passes through  $y = 3$  and  $y = -3$ . Also, when  $y = 0$ , then  $x = 2$  and  $x = -2$ . We drew the base of  $S$  below. We slice  $S$  using slices that are parallel to the  $yz$ -plane. (That is, we chop the  $x$ -axis from  $x = -2$  to  $x = 2$ . We express everything in terms of  $x$ . So, for example, the thickness of the slice that we study is  $dx$ .) In particular, we focus on the slice of  $S$  that lives above the indicated line segment on the base of  $S$ . The slice of  $S$  above the line segment is drawn in the middle picture. This slice is a triangle with thickness  $dx$ . The volume of the slice is  $\frac{1}{2} \cdot \text{base} \cdot \text{ht} \cdot \text{thickness}$ . We draw one more picture to learn that the height of our slice is exactly equal to  $\frac{1}{2} \cdot \text{base}$ .



The slice above the indicated line segment looks like



The slice with  $x$ -coordinate  $x$  has volume

$$\frac{1}{2} \cdot \text{base} \cdot \text{ht} \cdot \text{thickness} = \frac{1}{2} \cdot \text{base} \cdot \frac{1}{2} \cdot \text{base} \cdot dx = \frac{1}{4} \text{base}^2 dx.$$

Of course, the base is the big  $y$ -coordinate minus the little  $y$ -coordinate or  $2\sqrt{9 - \frac{9}{4}x^2}$ . The slice with  $x$ -coordinate  $x$  has volume

$$\frac{1}{4} \text{base}^2 dx = \frac{1}{4} \left( 2\sqrt{9 - \frac{9}{4}x^2} \right)^2 dx = \frac{1}{4} (4(9 - \frac{9}{4}x^2)) dx = (9 - \frac{9}{4}x^2) dx.$$

The volume of the solid is

$$\int_{-2}^2 (9 - \frac{9}{4}x^2) dx = (9x - \frac{9}{4} \frac{x^3}{3}) \Big|_{-2}^2 = 2(18 - \frac{3 \cdot 8}{4}) = 2(18 - 6) = \boxed{24}.$$

3. (6 points) Consider the sequence  $\{a_n\}$  with  $a_0 = 0$ , and for all  $n \geq 1$ ,  $a_n = \sqrt{6 + a_{n-1}}$ . Prove that this sequence is increasing. Prove that this sequence is bounded. Deduce that the sequence converges. Find the limit of the sequence.

We see that  $0 < \sqrt{6} < \sqrt{6 + \sqrt{6}}$ . So the series starts out to be increasing. Assume that after a while we have  $a_{n-1} < a_n$ . Then  $6 + a_{n-1} < 6 + a_n$  and  $\sqrt{6 + a_{n-1}} < \sqrt{6 + a_n}$ . But  $\sqrt{6 + a_{n-1}} = a_n$  and  $\sqrt{6 + a_n} = a_{n+1}$ . In other words, we have shown that: if  $a_{n-1}$  is less than  $a_n$ , then  $a_n$  is also less than  $a_{n+1}$ . So this sequence keeps on increasing forever. (This technique is called Mathematical Induction!)

We also use Mathematical Induction to show that 6 is an upper bound for this sequence. We have  $0 < 6$ , and  $\sqrt{6} < 6$ . Suppose  $a_{n-1} < 6$ , then  $6 + a_{n-1} < 6 + 6$  and  $\sqrt{6 + a_{n-1}} < \sqrt{12}$ . But  $\sqrt{6 + a_{n-1}}$  is also called  $a_n$  and  $\sqrt{12} < 6$ . We have shown that if  $a_{n-1}$  is less than 6, then  $a_n$  is also less than 6. So this sequence keeps on being less than 6 forever!

Our sequence is an increasing bounded sequence. The Completeness Axiom guarantees that our sequence has a limit. Let  $L = \lim_{n \rightarrow \infty} a_n$ . We may take

$\lim_{n \rightarrow \infty}$  of both sides of  $a_n = \sqrt{6 + a_{n-1}}$  to see that  $\lim_{n \rightarrow \infty} a_n = \sqrt{6 + \lim_{n \rightarrow \infty} a_{n-1}}$ ; so,  $L = \sqrt{6 + L}$  or  $L^2 = 6 + L$ . Thus,  $L^2 - L - 6 = 0$  or  $(L - 3)(L + 2) = 0$ . So,  $L = 3$  or  $L = -2$  (But  $L$  can not be  $-2$  because every element of the sequence is at least 0.) Thus,  $\boxed{L = 3}$ .

4. (6 points) A ball is dropped from a height of 100 feet. Each time it hits the ground, it rebounds to  $\frac{2}{3}$  its previous height. Find the total distance the ball travels before coming to rest.

The most interesting part of the problem is that the first event is only down. Every other even is a round trip: up and down. The distance the ball travels is

$$\begin{aligned} & 100 + \frac{2}{3}(100) + \frac{2}{3}(100) + \left(\frac{2}{3}\right)^2(100) + \left(\frac{2}{3}\right)^2(100) + \left(\frac{2}{3}\right)^3(100) + \left(\frac{2}{3}\right)^3(100) + \dots \\ & = 100 + \frac{2}{3}(200)\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots\right) = \boxed{100 + \frac{2}{3}(200)\frac{1}{1-\frac{2}{3}}} \end{aligned}$$

We used the fact that the geometric series with initial term  $a$  and radius  $r$  converges to  $\frac{a}{1-r}$ , provided  $-1 < r < 1$ .

5. (6 points) Let  $f(x)$  be the power series  $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2n+1}$ . Where does  $f(x)$  converge?

It is clear that  $f(3)$  converges. Fix some number  $x$ , with  $x \neq 3$ . We use the ratio test. Let

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-3)^{n+1}}{2(n+1)+1}}{\frac{(x-3)^n}{2n+1}} \right| = \lim_{n \rightarrow \infty} \frac{|x-3|^{n+1}(2n+1)}{|x-3|^n(2(n+1)+1)} = \lim_{n \rightarrow \infty} \frac{|x-3|(2n+1)}{(2n+3)} \\ &= \lim_{n \rightarrow \infty} \frac{|x-3|(2+\frac{1}{n})}{(2+\frac{3}{n})} = |x-3|. \end{aligned}$$

If  $\rho < 1$ , then  $f(x)$  converges. If  $1 < \rho$ , then  $f(x)$  diverges. If  $\rho = 1$ , then we need to do something else. Well,  $\rho < 1$  for  $|x-3| < 1$ ; that is,  $-1 < x-3 < 1$  or  $2 < x < 4$ . (So if  $2 < x < 4$ , then  $f(x)$  converges.) We see that  $1 < \rho$  when  $1 < |x-3|$ . In other words,  $1 < \rho$  when  $1 < x-3$ , that is:  $4 < x$ ; also,  $\rho < 1$  when  $x-3 < -1$ , that is:  $x < 2$ . (So, if  $x < 2$  or  $4 < x$ , then  $f(x)$  converges.)

We now study  $f(x)$  for  $x = 2$  and for  $x = 4$ . We see that  $f(4) = \sum_{n=0}^{\infty} \frac{1}{2n+1}$ . We compare  $f(4)$  to the Harmonic series using the limit comparison test. We see that

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \lim_{n \rightarrow \infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$$

This limit is a number. This limit is not zero or infinity. Thus,  $f(4)$  and the Harmonic series both converge or both diverge. The Harmonic series diverges, thus,  $f(4)$  diverges. We see that

$$f(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \dots$$

This is an alternating series. The terms, in absolute value, decrease to zero. The Alternating Series Test applies. We conclude that  $f(2)$  converges. Thus,

$f(x)$  converges for  $2 \leq x < 4$  and  $f(x)$  diverges everywhere else.

6. (6 points) Find the second Taylor Polynomial  $T_2(x)$  for  $f(x) = \sqrt{x}$  about  $a = 4$ . Give an upper bound for the error that is introduced if  $T_2(x)$  is used to approximate  $f(x)$  for  $4 \leq x \leq 4.2$ .

We compute

$$\begin{aligned} f(x) &= \sqrt{x} & f(4) &= 2 \\ f'(x) &= \frac{1}{2\sqrt{x}} & f'(4) &= \frac{1}{4} \\ f''(x) &= \frac{-1}{2 \cdot 2(x)^{3/2}} & f''(4) &= \frac{-1}{32} \\ f'''(x) &= \frac{3}{2 \cdot 2 \cdot 2(x)^{5/2}}. \end{aligned}$$

We know that

$$T_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2.$$

In other words,

$$T_2(x) = 2 + \frac{1}{4}(x-4) + \frac{-1}{2 \cdot 32}(x-4)^2.$$

Taylor's Theorem says that

$$|f(x) - T_2(x)| = |R_2(x)| = \left| \frac{f^3(c)}{3!}(x-4)^3 \right|$$

for some  $c$  between  $x$  and  $4$ . For us

$$|f^3(c)| = \frac{3}{8(c)^{5/2}}$$

and  $4 \leq c \leq 4.2$ . The biggest possible value of  $f^3(c)$  occurs when  $c$  takes on the least value. (The fraction is largest when the denominator is smallest.) So,  $|f^3(c)| \leq \frac{3}{8(4)^{5/2}} = \frac{3}{8 \cdot 32}$ . We have

$$|f(x) - T_2(x)| = \left| \frac{f^3(c)}{3!}(x-4)^3 \right| \leq \frac{3}{3!8 \cdot 32} (.2)^3$$

for  $4 \leq x \leq 4.2$ . We conclude that

$$T_2(x) = 2 + \frac{1}{4}(x-4) + \frac{-1}{2 \cdot 32}(x-4)^2 \text{ approximates } f(x) = \sqrt{4} \text{ with an error at most } \frac{3}{3!8 \cdot 32} (.2)^3 \text{ for } 4 \leq x \leq 4.2$$

7. (7 points) Find  $\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$ .

We see that

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) - x + \frac{1}{6}x^3}{x^5} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5} = \lim_{x \rightarrow 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \dots\right) = \boxed{\frac{1}{5!}}. \end{aligned}$$

8. (7 points) Approximate  $\int_0^1 x \cos(x^3) dx$  with an error at most  $10^{-3}$ .  
Justify your answer very thoroughly.

We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

It follows that

$$\cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots$$

Thus,

$$\begin{aligned} \int_0^1 x \cos(x^3) dx &= \int_0^1 x \left( 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots \right) dx \\ &= \int_0^1 \left( x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \dots \right) dx \\ &= \left( \frac{x^2}{2} - \frac{x^8}{2! \cdot 8} + \frac{x^{14}}{4! \cdot 14} - \frac{x^{20}}{6! \cdot 20} + \dots \right) \Big|_0^1 \\ &= \frac{1}{2} - \frac{1}{2! \cdot 8} + \frac{1}{4! \cdot 14} - \frac{1}{6! \cdot 20} + \dots \end{aligned}$$

The above series is an alternating series. The terms, in absolute value, go to zero. The Alternating Series Test applies. We see that  $\frac{1}{6! \cdot 20} < 10^{-3}$ ; so the alternating series test says that

$$\left| \int_0^1 x \cos(x^3) dx - \left( \frac{1}{2} - \frac{1}{2! \cdot 8} + \frac{1}{4! \cdot 14} \right) \right| < \frac{1}{6! \cdot 20} < 10^{-3}.$$

Thus,

$\frac{1}{2} - \frac{1}{2! \cdot 8} + \frac{1}{4! \cdot 14} \text{ approximates } \int_0^1 x \cos(x^3) dx \text{ with an error at most } 10^{-3}.$
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