

Math 142, Exam 2, Spring 2011 Solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it - I will still grade your exam.

The exam is worth 50 points. **SHOW** your work. CIRCLE your answer. **CHECK** your answer whenever possible.

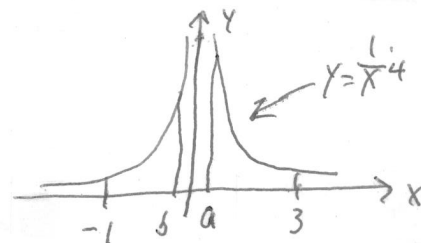
**No Calculators or Cell phones.** I will post the solutions on my website shortly after the exam is finished.

1. (5 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let  $f(x)$  be a function defined on the closed interval  $a \leq x \leq b$ . For each partition  $P$  of the closed interval  $[a, b]$  (so,  $P$  is  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ ), let  $\Delta_i = x_i - x_{i-1}$ , and pick  $x_i^* \in [x_{i-1}, x_i]$ . The definite integral  $\int_a^b f(x)dx$  is the limit over all partitions  $P$  as all  $\Delta_i$  go to zero of  $\sum_{i=1}^n f(x_i^*)\Delta_i$ .

2. (5 points) ) **Compute**  $\int_{-1}^3 \frac{1}{x^4} dx$ .

The function  $f(x) = \frac{1}{x^4}$  goes to infinity as  $x$  gets near 0:



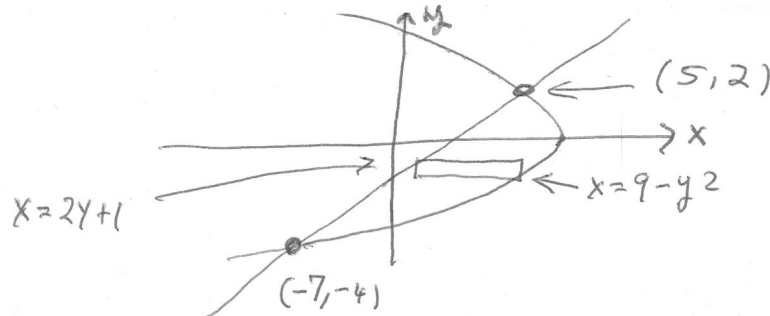
The answer will either be a positive number or  $+\infty$ . The integral is equal to

$$\begin{aligned} & \lim_{b \rightarrow 0^-} \int_{-1}^b \frac{1}{x^4} dx + \lim_{a \rightarrow 0^+} \int_a^3 \frac{1}{x^4} dx \\ &= \lim_{b \rightarrow 0^-} \frac{1}{-3x^3} \Big|_{-1}^b + \lim_{a \rightarrow 0^+} \frac{1}{-3x^3} \Big|_a^3 \\ &= \lim_{b \rightarrow 0^-} \frac{1}{-3b^3} - \frac{1}{3} + \lim_{a \rightarrow 0^+} \frac{1}{-81} + \frac{1}{3a^3} \end{aligned}$$

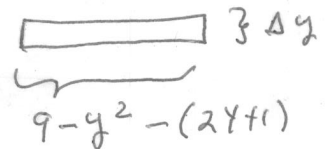
Both limits  $\lim_{b \rightarrow 0^-} \frac{1}{-3b^3}$  and  $\lim_{a \rightarrow 0^+} \frac{1}{3a^3}$  equal  $+\infty$ . The answer is  $+\infty$ .

3. (5 points) Find the area between  $x + y^2 = 9$  and  $2y - x + 1 = 0$ .

The intersection occurs when  $(2y+1)+y^2 = 9$ . This is the same as  $y^2+2y-8 = 0$ ; or  $(y+4)(y-2) = 0$ ; or  $y = 2, -4$ . When  $y = 2$ , we have  $x = 5$  (on both graphs). When  $y = -4$ , we have  $x = -7$  (on both graphs). The picture is



We use rectangles that are short and wide. That is we partition the  $y$ -axis from  $y = -4$  to  $y = 2$ . For each small interval of the  $y$ -axis we draw a rectangle:



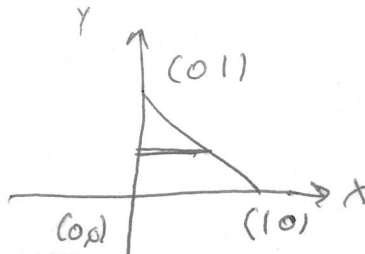
of area  $[9 - y^2 - (2y + 1)]\Delta y$ . The area of the entire region is

$$\int_{-4}^2 [9 - y^2 - (2y + 1)] dy = \int_{-4}^2 [8 - y^2 - 2y] dy = \left( 8y - \frac{y^3}{3} - y^2 \right) \Big|_{-4}^2$$

$$= \left[ 8(2) - \frac{2^3}{3} - 2^2 - \left( 8(-4) - \frac{(-4)^3}{3} - (-4)^2 \right) \right]$$

4. (5 points) Consider a solid whose base is the triangular region with vertices  $(0,0)$ ,  $(1,0)$ , and  $(0,1)$ . The cross sections of this solid perpendicular to the  $y$ -axis are equilateral triangles. Find the volume of the solid.

The base of the solid is



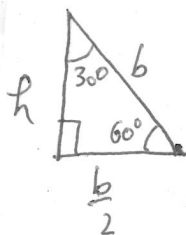
We partition the  $y$ -axis from  $y = 0$  to  $y = 1$ . We consider the slice above each small interval of the  $y$ -axis



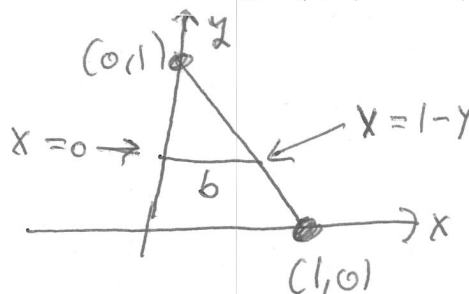
This slice has volume equal to the area of the face times the thickness. The volume of the slice is  $\frac{1}{2}bht$  where  $b$  is the base of the triangle and  $h$  is the height of the triangle.



The triangle is equilateral so  $h = \frac{\sqrt{3}}{2}b$ .



The volume of the slice is  $\frac{1}{2}bht = \frac{1}{2} \frac{\sqrt{3}}{2} b^2 \Delta y$ . We can express  $b$  in terms of  $y$  by looking at the original picture of the base. The line through  $(1,0)$  and  $(0,1)$  is  $x + y = 1$ . We see that  $b = 1 - y$ :



The volume of the slice is  $= \frac{1}{2} \frac{\sqrt{3}}{2} b^2 \Delta y = \frac{\sqrt{3}}{4} (1 - y)^2 \Delta y$ . The volume of the solid is

$$\frac{\sqrt{3}}{4} \int_0^1 (1 - y)^2 dy = -\frac{\sqrt{3}}{4} \frac{(1 - y)^3}{3} \Big|_0^1 = \boxed{\frac{\sqrt{3}}{12}}$$

5. (6 points)) **Find**  $\int \sec x \tan^2 x dx$ . **Check your answer.**

We do this integral by parts. Let  $u = \tan x$  and  $dv = \sec x \tan x dx$ . We compute  $du = \sec^2 x dx$  and  $v = \sec x$ . We have

$$\begin{aligned} \int \sec x \tan^2 x dx &= \sec x \tan x - \int \sec^3 x dx = \sec x \tan x - \int \sec x (\tan^2 x + 1) dx \\ &= \sec x \tan x - \int \sec x \tan^2 x dx - \int \sec x dx. \end{aligned}$$

Add  $\int \sec x \tan^2 x dx$  to both sides

$$2 \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x dx = \sec x \tan x - \ln |\sec x + \tan x| + C.$$

Divide by 2 to see that

$$\boxed{\int \sec x \tan^2 x dx = \frac{1}{2}(\sec x \tan x - \ln |\sec x + \tan x|) + C.}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} & \frac{1}{2} \left( \sec x \sec^2 x + \sec x \tan x \tan x - \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \right) \\ &= \frac{1}{2} (\sec x \sec^2 x + \sec x \tan x \tan x - \sec x) \\ &= \frac{1}{2} \sec x (\sec^2 x + \tan^2 x - 1) = \frac{1}{2} \sec x (\tan^2 x + \tan^2 x). \checkmark \end{aligned}$$

6. (6 points) ) **Find**  $\int \frac{x^3 - 3x^2 - x + 5}{x^3(x-1)} dx$ . **Check your answer.**

We first solve

$$\frac{x^3 - 3x^2 - x + 5}{x^3(x-1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-1}.$$

Multiply both sides by  $x^3(x-1)$  to obtain

$$\begin{aligned} x^3 - 3x^2 - x + 5 &= A(x-1) + Bx(x-1) + Cx^2(x-1) + Dx^3. \\ &= (C+D)x^3 + (B-C)x^2 + (A-B)x - A. \end{aligned}$$

Equate the corresponding coefficients to see that

$$1 = C + D, \quad -3 = B - C, \quad -1 = A - B, \quad 5 = -A.$$

Thus,  $A = -5$ ,  $B = -4$ ,  $C = -1$ , and  $D = 2$ . We verify

$$\begin{aligned} \frac{-5}{x^3} + \frac{-4}{x^2} + \frac{-1}{x} + \frac{2}{x-1} &= \frac{-5(x-1) - 4x(x-1) - x^2(x-1) + 2x^3}{x^3(x-1)} \\ &= \frac{x^3 - 3x^2 - x + 5}{x^3(x-1)}. \end{aligned}$$

We compute the integral

$$\int \frac{-5}{x^3} + \frac{-4}{x^2} + \frac{-1}{x} + \frac{2}{x-1} dx = \boxed{\frac{5}{2x^2} + \frac{4}{x} - \ln|x| + 2\ln|x-1| + C}$$

7. (6 points) ) **Find**  $\int \frac{x+4}{x^2+2x+5} dx$ . **Check your answer.**

$$\begin{aligned} \int \frac{x+4}{x^2+2x+5} dx &= \int \frac{x+1}{x^2+2x+5} dx + \int \frac{3}{x^2+2x+5} dx \\ &= \frac{1}{2} \ln|x^2+2x+5| + \int \frac{3}{(x+1)^2+4} dx \\ &= \boxed{\frac{1}{2} \ln|x^2+2x+5| + \frac{3}{2} \arctan\left(\frac{x+1}{2}\right) + C} \end{aligned}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} &\frac{x+1}{x^2+2x+5} + \frac{3}{2} \frac{\frac{1}{2}}{\left(\frac{x+1}{2}\right)^2+1} \\ &= \frac{x+1}{x^2+2x+5} + \frac{3}{4\left(\frac{(x+1)^2}{4}+1\right)} \\ &= \frac{x+1}{x^2+2x+5} + \frac{3}{x^2+2x+5} \checkmark \end{aligned}$$

8. (6 points) ) **Find**  $\int \frac{1}{x^2\sqrt{25-x^2}} dx$ . **Check your answer.**

Let  $x = 5 \sin \theta$ . It follows that  $dx = 5 \cos \theta d\theta$  and

$$25 - x^2 = 25 - (5 \sin \theta)^2 = 25(1 - \sin^2 \theta) = 25 \cos^2 \theta.$$

The integral is

$$\int \frac{5 \cos \theta d\theta}{(25 \sin^2 \theta) 5 \cos \theta} = \frac{1}{25} \int \csc^2 \theta d\theta = \frac{-1}{25} \cot \theta + C = \boxed{\frac{-\sqrt{25-x^2}}{25x} + C}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} \frac{25x \frac{2x}{2\sqrt{25-x^2}} + 25\sqrt{25-x^2}}{(25x)^2} &= \frac{\frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}}{25x^2} = \frac{x^2 + (25-x^2)}{25x^2\sqrt{25-x^2}} \\ &= \frac{25}{25x^2\sqrt{25-x^2}} \checkmark \end{aligned}$$

9. (6 points) ) **Find**  $\int \frac{dx}{\sqrt{x^2-9}}$  . **Check your answer.**

Let  $x = 3 \sec \theta$  . It follows that  $dx = 3 \sec \theta \tan \theta d\theta$  and

$$x^2 - 9 = (3 \sec \theta)^2 - 9 = 9(1 - \sec^2 \theta) = 9 \tan^2 \theta.$$

The integral is equal to

$$\begin{aligned} \int \frac{3 \sec \theta \tan \theta d\theta}{3 \tan \theta} &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| + C \\ &= \ln \left| \frac{x + \sqrt{x^2-9}}{3} \right| + C = \ln |x + \sqrt{x^2-9}| - \ln 3 + C = \boxed{\ln |x + \sqrt{x^2-9}| + K} \end{aligned}$$

where  $K$  is the constant  $-\ln 3 + C$  .

Check . The derivative of the proposed answer is

$$\frac{1 + \frac{2x}{2\sqrt{x^2-9}}}{x + \sqrt{x^2-9}} = \frac{1 + \frac{x}{\sqrt{x^2-9}}}{x + \sqrt{x^2-9}}.$$

Multiply top and bottom to see that this expression is equal to

$$= \frac{\sqrt{x^2-9} + x}{\sqrt{x^2-9}(x + \sqrt{x^2-9})} \cdot \sqrt{x^2-9}$$