Math 142, Exam 2, Spring 2011 Solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible.

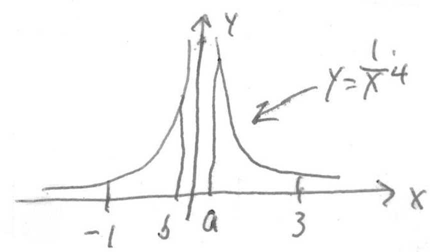
No Calculators or Cell phones. I will post the solutions on my website shortly after the exam is finished.

1. (5 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (5 points)) **Compute** $\int_{-1}^{3} \frac{1}{x^4} dx$.

The function $f(x) = \frac{1}{x^4}$ goes to infinity as x gets near 0:



The answer will either be a positive number or $+\infty$. The integral is equal to

$$\lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^4} dx + \lim_{a\to 0^+} \int_{a}^{3} \frac{1}{x^4} dx$$

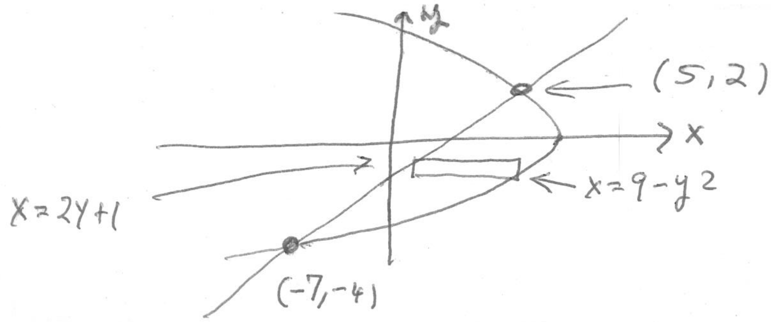
$$= \lim_{b\to 0^-} \left. \frac{1}{-3x^3} \right|_{-1}^{b} + \lim_{a\to 0^+} \left. \frac{1}{-3x^3} \right|_{a}^{3}$$

$$= \lim_{b\to 0^-} \frac{1}{-3b^3} - \frac{1}{3} + \lim_{a\to 0^+} \frac{1}{-81} + \frac{1}{3a^3}$$

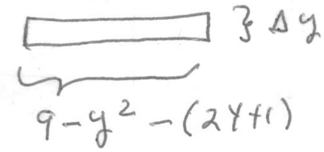
Both limits $\lim_{b\to 0^-} \frac{1}{-3b^3}$ and $\lim_{a\to 0^+} \frac{1}{3a^3}$ equal $+\infty$. The answer is $\boxed{+\infty}$.

3. (5 points)) **Find the area between $x + y^2 = 9$ and $2y - x + 1 = 0$.**

The intersection occurs when $(2y+1)+y^2 = 9$. This is the same as $y^2+2y-8 = 0$; or $(y+4)(y-2) = 0$; or $y = 2, -4$. When $y = 2$, we have $x = 5$ (on both graphs). When $y = -4$, we have $x = -7$ (on both graphs). The picture is



We use rectangles that are short and wide. That is we partition the y-axis from $y = -4$ to $y = 2$. For each small interval of the y-axis we draw a rectangle:



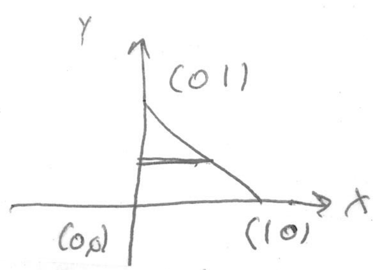
of area $[9 - y^2 - (2y + 1)]\Delta y$. The area of the entire region is

$$\int_{-4}^{2} [9 - y^2 - (2y + 1)]dy = \int_{-4}^{2} [8 - y^2 - 2y]dy = \left(8y - \frac{y^3}{3} - y^2\right)\Bigg|_{-4}^{2}$$

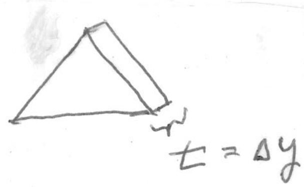
$$= \boxed{8(2) - \frac{2^3}{3} - 2^2 - \left(8(-4) - \frac{(-4)^3}{3} - (-4)^2\right)}$$

4. (5 points)) **Consider a solid whose base is the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. The cross sections of this solid perpendicular to the y-axis are equilateral triangles. Find the volume of the solid.**

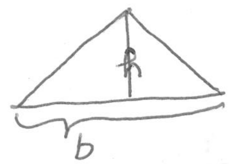
The base of the solid is



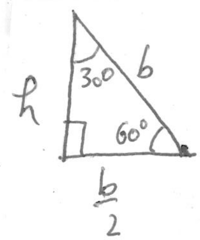
We partition the y-axis from $y = 0$ to $y = 1$. We consider the slice above each small interval of the y-axis



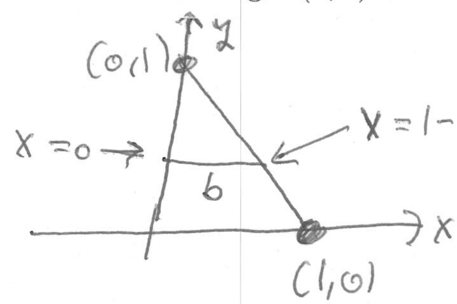
This slice has volume equal to the area of the face times the thickness. The volume of the slice is $\frac{1}{2}bht$ where b is the base of the triangle and h is the height of the triangle.



The triangle is equilateral so $h = \frac{\sqrt{3}}{2}b$.



The volume of the slice is $\frac{1}{2}bht = \frac{1}{2}\frac{\sqrt{3}}{2}b^2\Delta y$. We can express b in terms of y by looking at the original picture of the base. The line through $(1,0)$ and $(0,1)$ is $x+y=1$. We see that $b = 1-y$:



The volume of the slice is $= \frac{1}{2}\frac{\sqrt{3}}{2}b^2\Delta y = \frac{\sqrt{3}}{4}(1-y)^2\Delta y$. The volume of the solid is

$$\frac{\sqrt{3}}{4}\int_0^1 (1-y)^2 dy = -\frac{\sqrt{3}}{4}\frac{(1-y)^3}{3}\Bigg|_0^1 = \boxed{\frac{\sqrt{3}}{12}}.$$

5. (6 points)) **Find** $\int \sec x \tan^2 x\, dx$ **. Check your answer.**

We do this integral by parts. Let $u = \tan x$ and $dv = \sec x \tan x dx$. We compute $du = \sec^2 x dx$ and $v = \sec x$. We have

$$\int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec^3 x dx = \sec x \tan x - \int \sec x(\tan^2 x + 1)dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x dx - \int \sec x dx.$$

Add $\int \sec x \tan^2 x dx$ to both sides

$$2\int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x dx = \sec x \tan x - \ln|\sec x + \tan x| + C.$$

Divide by 2 to see that

$$\boxed{\int \sec x \tan^2 x\, dx = \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C.}$$

Check. The derivative of the proposed answer is

$$\frac{1}{2}\left(\sec x \sec^2 x + \sec x \tan x \tan x - \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\right)$$

$$= \frac{1}{2}\left(\sec x \sec^2 x + \sec x \tan x \tan x - \sec x\right)$$

$$= \frac{1}{2}\sec x\left(\sec^2 x + \tan^2 x - 1\right) = \frac{1}{2}\sec x\left(\tan^2 x + \tan^2 x\right). \checkmark$$

6. (6 points)) **Find** $\int \frac{x^3-3x^2-x+5}{x^3(x-1)}\, dx$. **Check your answer.**

We first solve

$$\frac{x^3-3x^2-x+5}{x^3(x-1)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-1}.$$

Multiply both sides by $x^3(x-1)$ to obtain

$$x^3 - 3x^2 - x + 5 = A(x-1) + Bx(x-1) + Cx^2(x-1) + Dx^3.$$

$$= (C+D)x^3 + (B-C)x^2 + (A-B)x - A.$$

Equate the corresponding coefficientsd to see that

$$1 = C + D, \quad -3 = B - C, \quad -1 = A - B, \quad 5 = -A.$$

Thus, $A = -5$, $B = -4$, $C = -1$, and $D = 2$. We verify

$$\frac{-5}{x^3} + \frac{-4}{x^2} + \frac{-1}{x} + \frac{2}{x-1} = \frac{-5(x-1) - 4x(x-1) - x^2(x-1) + 2x^3}{x^3(x-1)}$$

$$= \frac{x^3 - 3x^2 - x + 5}{x^3(x-1)}.$$

We compute the integral

$$\int \frac{-5}{x^3} + \frac{-4}{x^2} + \frac{-1}{x} + \frac{2}{x-1} dx = \boxed{\frac{5}{2x^2} + \frac{4}{x} - \ln|x| + 2\ln|x-1| + C}.$$

7. (6 points)) **Find** $\int \frac{x+4}{x^2+2x+5}\, dx$ **. Check your answer.**

$$\begin{aligned}\int \frac{x+4}{x^2+2x+5}\, dx &= \int \frac{x+1}{x^2+2x+5}\, dx + \int \frac{3}{x^2+2x+5}\, dx\\ &= \frac{1}{2}\ln|x^2+2x+5| + \int \frac{3}{(x+1)^2+4}\, dx\\ &= \boxed{\frac{1}{2}\ln|x^2+2x+5| + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + C}\end{aligned}$$

Check . The derivative of the proposed answer is

$$\begin{aligned}&\frac{x+1}{x^2+2x+5} + \frac{3}{2}\frac{\frac{1}{2}}{\left(\frac{x+1}{2}\right)^2+1}\\ &= \frac{x+1}{x^2+2x+5} + \frac{3}{4\left(\frac{(x+1)^2}{4}+1\right)}\\ &= \frac{x+1}{x^2+2x+5} + \frac{3}{x^2+2x+5} \checkmark\end{aligned}$$

8. (6 points)) **Find** $\int \frac{1}{x^2\sqrt{25-x^2}}\, dx$ **. Check your answer.**

Let $x = 5\sin\theta$. It follows that $dx = 5\cos\theta d\theta$ and

$$25 - x^2 = 25 - (5\sin\theta)^2 = 25(1-\sin^2\theta) = 25\cos^2\theta.$$

The integral is

$$\int \frac{5\cos\theta d\theta}{(25\sin^2\theta)5\cos\theta} = \frac{1}{25}\int \csc^2\theta d\theta = \frac{-1}{25}\cot\theta + C = \boxed{\frac{-\sqrt{25-x^2}}{25x} + C}$$

Check . The derivative of the proposed answer is

$$\frac{25x\frac{2x}{2\sqrt{25-x^2}} + 25\sqrt{25-x^2}}{(25x)^2} = \frac{\frac{x^2}{\sqrt{25-x^2}} + \sqrt{25-x^2}}{25x^2} = \frac{x^2 + (25-x^2)}{25x^2\sqrt{25-x^2}}$$

$$= \frac{25}{25x^2\sqrt{25-x^2}}. \checkmark$$

9. (6 points)) **Find $\int \frac{dx}{\sqrt{x^2-9}}$. Check your answer.**

Let $x = 3\sec\theta$. It follows that $dx = 3\sec\theta\tan\theta d\theta$ and

$$x^2 - 9 = (3\sec\theta) - 9 = 9(1 - \sec^2\theta) = 9\tan^2\theta.$$

The integral is equal to

$$\int \frac{3\sec\theta\tan\theta d\theta}{3\tan\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

$$= \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C = \ln\left|x + \sqrt{x^2-9}\right| - \ln 3 + C = \boxed{\ln\left|x+\sqrt{x^2-9}\right| + K}$$

where K is the constant $-\ln 3 + C$.

Check. The derivative of the proposed answer is

$$\frac{1 + \frac{2x}{2\sqrt{x^2-9}}}{x + \sqrt{x^2-9}} = \frac{1 + \frac{x}{\sqrt{x^2-9}}}{x+\sqrt{x^2-9}}.$$

Multiply top and bottom to see that this expression is equal to

$$= \frac{\sqrt{x^2-9} + x}{\sqrt{x^2-9}(x + \sqrt{x^2-9})}. \checkmark$$