

Math 142, Final Exam, Fall 2009

Write everything on the blank paper provided. You should **KEEP** this piece of paper. If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it - I will still grade your exam.

The exam is worth 200 points. There are **20** problems. Each problem is worth 10 points. **SHOW** your work. **CIRCLE** your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. Find $\int 3\sqrt{4+2x}dx$. Check your answer.

Let $u = 4 + 2x$; so, $du = 2dx$. The integral is

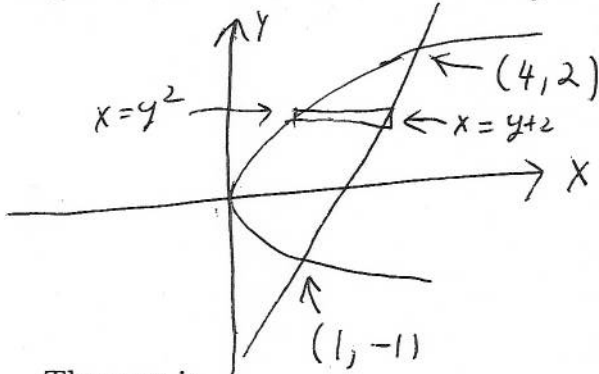
$$(3/2) \int u^{1/2} du = (3/2)(2/3)u^{3/2} + C = \boxed{(4+2x)^{3/2} + C}.$$

Check. The derivative of the proposed answer is $(3/2)(4+2x)^{1/2} \cdot 2 \checkmark$.

2. Define the definite integral. Give a complete definition. Be sure to explain all of your notation.

Let $f(x)$ be a function defined on the closed interval $a \leq x \leq b$. For each partition P of the closed interval $[a, b]$ (so, P is $a = x_0 \leq x_1 \leq \dots \leq x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^n f(x_i^*)\Delta_i$.

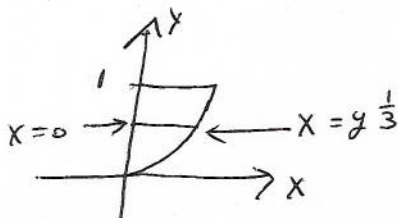
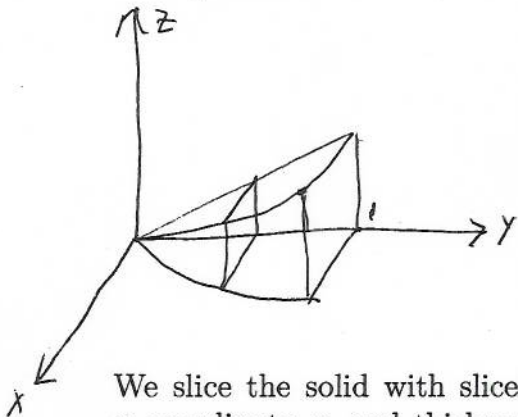
3. Find the area between $x = y^2$ and $y = x - 2$.



The area is

$$\int_{-1}^2 (y+2-y^2)dy = \left. \frac{y^2}{2} + 2y - \frac{y^3}{3} \right|_{-1}^2 = 2 + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) = \boxed{\frac{9}{2}}.$$

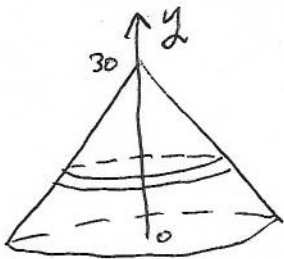
4. Find the volume of the solid whose base in the xy plane is the region bounded between the curve $y = x^3$ and the y -axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y -axis are squares.



We slice the solid with slices parallel to the xz -plane. The indicated slice has y -coordinate y and thickness dy . The volume of the indicated slice is $y^{2/3}dy$. The volume of the solid is

$$\int_0^1 y^{2/3} dy = \frac{3y^{5/3}}{5} \Big|_0^1 = \boxed{\frac{3}{5}}$$

5. A conical water tank whose base is a circle of radius 10 feet and whose height is 30 ft is filled with water to a depth of 15 feet. How much work is required to pump all of the water out through a hole in the top of the tank? (The density of water is 62.4 pounds per cubic foot.) Be sure to include units in your answer.



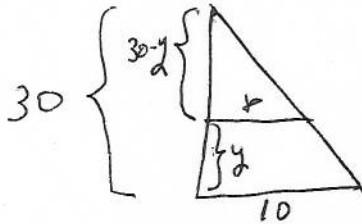
We find the work required to lift the layer of water whose y -coordinate is y and whose thickness is dy . The work to lift this layer is

the weight of the layer times the distance it must be lifted

= 62.4 times the volume of the layer times the distance it must be lifted

$$= 62.4\pi r^2 d dy,$$

where r is the radius of the layer and d is the distance the layer we lift the layer.



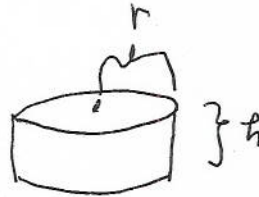
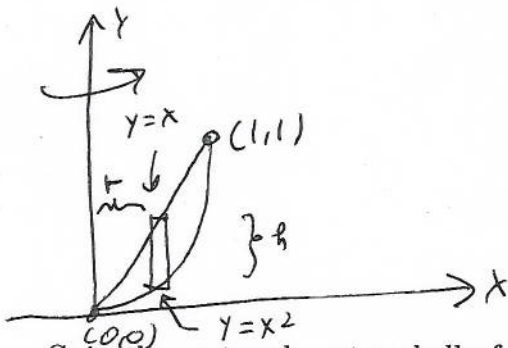
The distance we lift the layer of water is $d = 30 - y$. Similar triangles shows us that $\frac{r}{30-y} = \frac{10}{30}$; so, $r = \frac{1}{3}(30 - y)$. The work to lift this layer is

$$= 62.4\pi \left(\frac{1}{3}(30 - y)\right)^2 (30 - y) dy.$$

The work to lift all of the water out of the tank is

$$\frac{62.4\pi}{9} \int_0^{15} (30 - y)^3 dy = \frac{62.4\pi}{9} \frac{(30 - y)^4}{-4} \Big|_0^{15} = \boxed{\frac{62.4\pi}{36}(30^4 - 15^4)\text{ft-lbs.}}$$

6. Consider the region in the first quadrant bounded by $y = x$ and $y = x^2$. Rotate this region about the y -axis. Find the volume.



Spin the rectangle get a shell of volume $2\pi rht = 2\pi x(x - x^2)dx$. The volume of the solid is

$$2\pi \int_0^1 x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4}\right) \Big|_0^1 = \frac{2\pi}{12} = \boxed{\frac{\pi}{6}}$$

7. Find the length of $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$.

The arc length is

$$\begin{aligned} \int_1^2 \sqrt{1 + (dy/dx)^2} dx &= \int_1^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx = \int_1^2 \sqrt{1 + (9/4)x} dx \\ &= \frac{4(2)(1 + (9/4)x)^{3/2}}{9(3)} \Big|_1^2 = \boxed{\frac{8}{27} \left(\left(1 + \frac{9}{2}\right)^{3/2} - \left(1 + \frac{9}{4}\right)^{3/2} \right)} \end{aligned}$$

8. Find $\int xe^x dx$. Check your answer.

Try integration by parts. Let $u = x$ and $dv = e^x dx$. Calculate $du = dx$ and $v = e^x$. The given problem is

$$\int u dv = uv - \int v du = xe^x - \int e^x dx = \boxed{xe^x - e^x + C.}$$

Check. The derivative of the proposed answer is

$$xe^x + e^x - e^x \checkmark.$$

9. Find $\int \tan^2 x \sec^4 x dx$. Check your answer.

The integral is

$$\int \tan^2 x \sec^2 x \sec^2 x dx = \int \tan^2 x (\tan^2 x + 1) \sec^2 x dx.$$

Let $u = \tan x$. It follows that $du = \sec^2 x dx$. The integral is

$$\int (u^4 + u^2) du = \frac{u^5}{5} + \frac{u^3}{3} + C = \boxed{\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.}$$

Check. The derivative of the proposed answer is

$$\tan^4 x \sec^2 x + \tan^2 x \sec^2 x = \tan^2 x \sec^2 x (\tan^2 x + 1) = \tan^2 x \sec^4 x \checkmark.$$

10. Find $\int \frac{\sqrt{x^2-25}}{x} dx$. Check your answer.

Let $x = 5 \sec \theta$. It follows that $x^2 - 25 = 25 \sec^2 \theta - 25 = 25(\sec^2 \theta - 1) = 25 \tan^2 \theta$. It also follows that $dx = 5 \sec \theta \tan \theta d\theta$. The original integral is equal to

$$\begin{aligned} \int \frac{5 \tan \theta (5 \sec \theta \tan \theta d\theta)}{5 \sec \theta} &= 5 \int \tan^2 \theta d\theta = 5 \int (\sec^2 \theta - 1) d\theta = 5(\tan \theta - \theta) + C \\ &= 5 \left(\frac{\sqrt{x^2-25}}{5} - \operatorname{arcsec}(x/5) \right) + C \\ &= \boxed{\sqrt{x^2-25} - 5 \operatorname{arcsec}(x/5) + C.} \end{aligned}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} \frac{2x}{2\sqrt{x^2-25}} - \frac{(\frac{1}{5})5}{\frac{x}{5}\sqrt{\frac{x^2}{25}-1}} &= \frac{x^2}{x\sqrt{x^2-25}} - \frac{25(\frac{1}{5})5}{25(\frac{x}{5})\sqrt{\frac{x^2}{25}-1}} \\ &= \frac{x^2}{x\sqrt{x^2-25}} - \frac{25}{x\sqrt{x^2-25}} = \frac{\sqrt{x^2-25}}{x} \checkmark. \end{aligned}$$

11. Find $\int \frac{x+1}{(x-1)^2} dx$. Check your answer.

We solve

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Multiply both sides by $(x-1)^2$ to obtain

$$x+1 = A(x-1) + B.$$

Equate the corresponding coefficients:

$$1 = A \quad \text{and} \quad 1 = B - A.$$

So, $A = 1$ and $B = 2$. We check this before we go any further:

$$\frac{1}{x-1} + \frac{2}{(x-1)^2} = \frac{x-1+2}{(x-1)^2} = \frac{x+1}{(x-1)^2} \checkmark$$

So the original problem is the same as

$$\int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx = \boxed{\ln|x-1| - \frac{2}{x-1} + C.}$$

12. Find $\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$.

The integral is equal to

$$\lim_{b \rightarrow \infty} \int_0^b \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} -2e^{-\sqrt{x}} \Big|_0^b = \lim_{b \rightarrow \infty} -2e^{-\sqrt{b}} + 2 = \lim_{b \rightarrow \infty} \frac{-2}{e^{\sqrt{b}}} + 2 = \boxed{2}.$$

13. Find the limit of the sequence whose n^{th} term is $a_n = \frac{\sin(4n)}{n}$.

L'Hopital's rule has nothing to do with this problem. We have

$$0 = \lim_{n \rightarrow \infty} \frac{-1}{n} \leq \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0. \text{ Thus } \boxed{\lim_{n \rightarrow \infty} a_n = 0}.$$

14. Consider the series $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$. Find a closed formula for the n^{th} partial sum $s_n = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right)$. Be sure to answer the question I have asked!

We compute a partial sum of a telescoping series:

$$\begin{aligned} s_n &= \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1} \right) + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \boxed{1 - \frac{1}{n+1}}. \end{aligned}$$

15. Consider the series $\sum_{k=1}^{\infty} (23)^k$. Find a closed formula for the n^{th} partial sum $s_n = \sum_{k=1}^n (23)^k$. Be sure to answer the question I have asked!

We compute a partial sum of a geometric series:

$$\begin{aligned} 22s_n &= 23s_n - s_n = 23 \sum_{k=1}^n (23)^k - \sum_{k=1}^n (23)^k \\ &= (23)^2 + \cdots + (23)^{n+1} - ((23) + \cdots + (23)^n) = (23)^{n+1} - 23; \end{aligned}$$

so,

$$\boxed{s_n = \frac{(23)^{n+1} - 23}{22}}.$$

16. Does the series $\sum_{k=1}^{\infty} \frac{\sqrt{k} + 1}{k^2 + 2k}$ converge? Justify your answer.

We apply the limit comparison test. We compare the given series to $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$. We calculate

$$\lim_{k \rightarrow \infty} \frac{\frac{\sqrt{k} + 1}{k^2 + 2k}}{\frac{1}{k^{3/2}}} = \lim_{k \rightarrow \infty} \frac{k^2 + k^{3/2}}{k^2 + 2k} = \lim_{k \rightarrow \infty} \frac{1 + \frac{1}{k^{1/2}}}{1 + \frac{2}{k}} = 1.$$

Since 1 is a number and 1 is not zero or infinity, we may apply the limit comparison test. The series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is the p -series with $p = 3/2 > 1$. This series converges; therefore,

$$\sum_{k=1}^{\infty} \frac{\sqrt{k} + 1}{k^2 + 2k} \text{ also converges.}$$

17. Find the third Taylor polynomial for $f(x) = \sqrt{x}$ about $c = 1$.

We compute

$$\begin{aligned} f(x) &= \sqrt{x} & f(1) &= 1 \\ f'(x) &= \frac{1}{2}x^{-1/2} & f'(1) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{4}x^{-3/2} & f''(1) &= -\frac{1}{4} \\ f'''(x) &= \frac{3}{8}x^{-5/2} & f'''(1) &= -\frac{3}{8} \end{aligned}$$

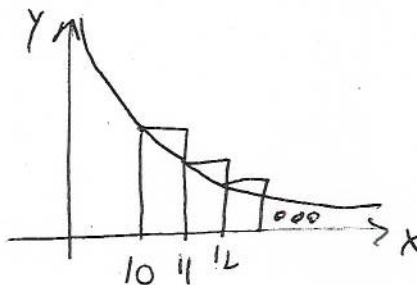
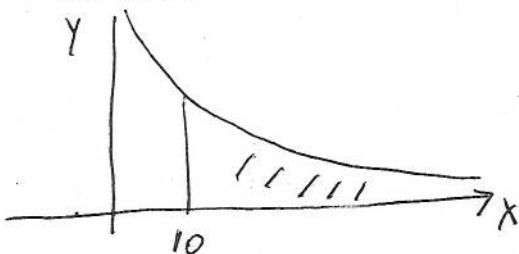
So,

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= 1 + \frac{1}{2}(x-1) + \frac{-1}{8}(x-1)^2 + \frac{3}{(8)(3!)}(x-1)^3.$$

18. Approximate $\sum_{n=10}^{\infty} \frac{1}{n^2}$. I want you to give both an under estimate and an over estimate. Explain what you are doing.

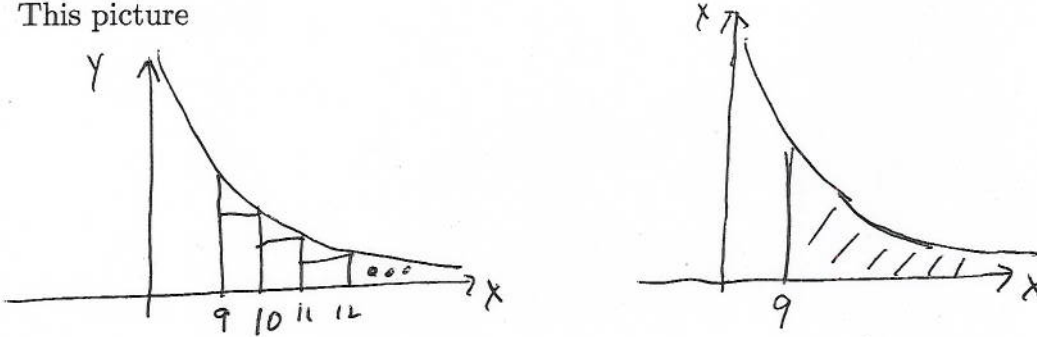
This picture



shows that

$$\int_{10}^{\infty} \frac{1}{x^2} dx \leq \sum_{n=10}^{\infty} \frac{1}{n^2}.$$

This picture



shows that

$$\sum_{n=10}^{\infty} \frac{1}{n^2} \leq \int_9^{\infty} \frac{1}{x^2} dx$$

We compute

$$\int_{10}^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left. \frac{-1}{x} \right|_{10}^b = \lim_{b \rightarrow \infty} \frac{-1}{b} + \frac{1}{10} = \frac{1}{10}.$$

We compute

$$\int_9^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left. \frac{-1}{x} \right|_9^b = \lim_{b \rightarrow \infty} \frac{-1}{b} + \frac{1}{9} = \frac{1}{9}.$$

We conclude that

$$\boxed{\frac{1}{10} \leq \sum_{n=10}^{\infty} \frac{1}{n^2} \leq \frac{1}{9}}.$$

19. Approximate $e^{-\frac{1}{100}}$ with an error at most 10^{-9} . Explain what you are doing

We know

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots,$$

for all x . So, in particular,

$$e^{-\frac{1}{100}} = 1 - \frac{1}{100} + \frac{1}{2(100)^2} - \frac{1}{3!(100)^3} + \frac{1}{4!(100)^4} - \frac{1}{5!(100)^5} + \dots$$

The alternating series test applies to the above series because the series alternates and the absolute value of the terms is decreasing to zero. We see that $\frac{1}{4!(100)^4} < (10)^{-9}$. We conclude that

$$\boxed{1 - \frac{1}{100} + \frac{1}{2(100)^2} - \frac{1}{3!(100)^3} \text{ approximates } e^{-\frac{1}{100}} \text{ with an error at most } 10^{-9}.}$$

20. What is the exact value of the sum $\sum_{k=1}^{\infty} \frac{1}{k(4^k)} = \frac{1}{4} + \frac{1}{2(4^2)} + \frac{1}{3(4^3)} + \frac{1}{4(4^4)} \dots$?

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$

for all x with $-1 < x < 1$. Integrate to see that

$$-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots,$$

for all x with $-1 < x < 1$. Plug in $x=0$ to see that $C=0$. So,

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots,$$

for $-1 < x < 1$. Plug in $x = \frac{1}{4}$ to learn

$$\boxed{-\ln \frac{3}{4}} = \frac{1}{4} + \frac{1}{2(4^2)} + \frac{1}{3(4^3)} + \frac{1}{4(4^4)} \dots$$