Quiz 9, September 20. 2016

Find $\int \frac{8dx}{(4x^2+1)^2}$.

Answer: Let $2x = \tan\theta$. It follows that $2dx = \sec^2\theta d\theta$ and $4x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. The original integral is

$$\int \frac{8(\frac{1}{2})\sec^2\theta d\theta}{\sec^4\theta} = \int \frac{4}{\sec^2\theta}d\theta = \int \cos^2\theta d\theta = 2\int(1+\cos 2\theta)d\theta = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$= \frac{2x}{\sqrt{4x^2+1}} + C = 2\left(\theta + \frac{2\sin(\theta)\cos(\theta)}{2}\right) + C.$$

Draw a right triangle with $2x = \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$. So this triangle has $2x$ on the opposite, 1 on the adjacent, and $\sqrt{4x^2+1}$ on the hypotenuse. So $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{2x}{\sqrt{4x^2+1}}$ and $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{4x^2+1}}$. Our integral is equal to

$$2\left(\theta + \frac{2\sin(\theta)\cos(\theta)}{2}\right) + C = 2\left(\arctan(2x) + \frac{2x}{4x^2+1}\right) + C = \boxed{2\arctan(2x) + \frac{4x}{4x^2+1} + C}.$$

Check: The derivative of the proposed answer is

$$\frac{2\cdot 2}{(2x)^2+1} - 4x\frac{8x}{(4x^2+1)^2} + \frac{4}{4x^2+1} = \frac{8}{4x^2+1} + \frac{-32x^2}{(4x^2+1)^2} = \frac{8(4x^2+1) - 32x^2}{(4x^2+1)^2}$$
$$= \frac{8dx}{(4x^2+1)^2}. \checkmark$$