Quiz 26, March 30, 2016

Where does the power series $f(x) = \sum_{n=0}^{\infty} \frac{nx^n}{n+2}$ converge? Justify your answer.

Answer: It is clear that $f(0)$ converges. We consider $x \neq 0$. We apply the ratio test. Let

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(n+1)x^{n+1}}{n+3}\right|}{\left|\frac{nx^n}{n+2}\right|} = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{n+3}\right| \left|\frac{n+2}{nx^n}\right| = \lim_{n\to\infty} \frac{(n+1)(n+2)}{(n+3)n} |x|.$$

Divide top and bottom by n^2 to see that

$$\rho = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{2}{n})}{(1+\frac{3}{n})} |x| = |x|.$$

If $\rho < 1$, then $f(x)$ converges. That is, if $-1 < x < 1$, then $f(x)$ converges.

If $1 < \rho$, then $f(x)$ diverges. That is, if $x < -1$ or $1 < x$, then $f(x)$ converges.

If $x = 1$, then $f(1) = \sum_{n=0}^{\infty} \frac{n}{n+2}$ which diverges by the individual term test for divergence, since $\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1 \neq 0$.

If $x = -1$ then $f(-1) = \sum_{n=0}^{\infty} \frac{n(-1)^n}{n+2}$ which diverges by the individual term test for divergence, since $\lim_{n\to\infty} \frac{n(-1)^n}{n+2}$ does not exist. (Half of the terms go to 1. The other half of the terms go to -1.)

We conclude that $f(x)$ converges for $-1 < x < 1$ and diverges everywhere else.