Quiz 13, February 17, 2016

Find $\int_0^2 \frac{1}{1-x^2}dx$.

Answer: The function $\frac{1}{1-x^2}$ has a veritical asymptote at $x=1$. We put a picture on another page. The integral is

$$\int_0^2 \frac{1}{1-x^2}dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x^2}dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{1-x^2}dx.$$

We see that $1-x^2=(1-x)(1+x)$. We find numbers A and B with

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}.$$

Multiply both sides by $1-x^2$ to see

$$1 = A(1+x)+B(1-x).$$

Plug in $x=-1$ to learn that $B=1/2$. Plug in $x=1$ to learn that $A=1/2$. We have shown that

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x}+\frac{1}{1+x}\right].$$

We verify before going any further. The right side is

$$\frac{1}{2}\frac{(1+x)+(1-x)}{(1-x)(1+x)} = \frac{1}{1-x^2}. \checkmark$$

Now we compute

$$\begin{aligned}
\int_0^2 \frac{1}{1-x^2}dx &= \lim_{b\to 1^-}\int_0^b \frac{1}{1-x^2}dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{1-x^2}dx \\
&= \lim_{b\to 1^-}\int_0^b \frac{1}{2}\left[\frac{1}{1-x}+\frac{1}{1+x}\right]dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{2}\left[\frac{1}{1-x}+\frac{1}{1+x}\right]dx \\
&= \lim_{b\to 1^-}\frac{1}{2}\left[-\ln|1-x|+\ln|1+x|\right]\Big|_0^b + \lim_{a\to 1^+}\frac{1}{2}\left[-\ln|1-x|+\ln|1+x|\right]\Big|_a^2 \\
&= \lim_{b\to 1^-}\frac{1}{2}\left[-\ln|1-b|+\ln|1+b|-(-\ln|1|+\ln|1|)\right] \\
&\quad + \lim_{a\to 1^+}\frac{1}{2}\left[-\ln|1-2|+\ln|1+2|-(-\ln|1-a|+\ln|1+a|)\right]|
\end{aligned}$$

We see that $\lim_{b\to 1^-}[-\ln|1-b|] = +\infty$ and $\lim_{a\to 1^+}\ln|1-a| = -\infty$.

We conclude that this integral diverges.