Math 142, Final Exam, Fall 2010

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.
The exam is worth 100 points. Each problem is worth 5 points. SHOW your work. $\boxed{CIRCLE}$ your answer.
No Calculators or Cell phones.

1. **Consider the region bounded by $y = x^2$ and $x = y - 6$. Revolve the region about $x = -4$. Find the volume of the resulting solid.**

2. **Consider a solid S. The base of S is an elliptical region with boundary curve $4x^2 + 9y^2 = 36$. The cross sections of S perpendicular to the x-axis are squares. Find the volume of S.**

3. **Consider the sequence $\{a_n\}$ with $a_0 = 0$, and for all $n \geq 1$, $a_n = \sqrt{20 + a_{n-1}}$. Prove that this sequence is increasing. Prove that this sequence is bounded. Deduce that the sequence converges. Find the limit of the sequence.**

4. **Suppose that the government pumps an extra \$1 billion into the economy. Assume that each business and individual saves 25% of its income and spends the rest, so that of the initial \$1 billion, 75% is respent by individuals and businesses. Of that amount, 75% is spent, and so forth. What is the total increase in spending due to the government action?**

5. **Let $f(x)$ be the power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n3^n}$. Where does $f(x)$ converge?**

6. **Find the second Taylor Polynomial $T_2(x)$ for $f(x) = \sqrt{x+1}$ about $a = 0$. Give an upper bound for the error that is introduced if $T_2(x)$ is used to approximate $f(x)$ for $0 \leq x \leq .2$.**

7. **Find $\lim_{x \to 0} \frac{\sin x^2 - x^2 + \frac{1}{6}x^6}{x^9}$.**

8. **Approximate** $\int_0^1 x\sin(x^3)dx$ **with an error at most** 10^{-3} **. Justify your answer very thoroughly.**

9. **Approximate** $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$ **with an error of at most** $\frac{1}{100}$ **. Explain what you are doing very thoroughly.**

10. **Does** $\sum\limits_{n=1}^{\infty} \frac{2n+3}{4n^2+5}$ **converge? Justify your answer very thoroughly.**

11. **Find** $\int_0^3 \frac{1}{(x-1)^2}dx$ **.**

12. **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a continuous function defined on the closed interval $[a,b]$. For each partition P of $[a,b]$ of the form $a = x_0 \le x_1 \le \cdots \le x_n = b$, let M_i be the maximum value of $f(x)$ on the subinterval $[x_{i-1}, x_i]$ and let m_i be the minimum value of $f(x)$ on $[x_{i-1}, x_i]$. If there is exactly one number with

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \text{this number} \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

as P varies over all partitions of $[a,b]$, then this number is called **the definite integral** of f on $[a,b]$ and this number is denoted $\int_a^b f(x)dx$.

13. **Find** $\int e^{2x}\cos x dx$ **. Check your answer.**

14. **Find** $\int \frac{3x^2-x+3}{(x-2)(x^2+9)}dx$ **. Check your answer.**

Use partial fractions to write

$$\frac{3x^2-x+3}{(x-2)(x^2+9)} = \frac{1}{x-2} + \frac{2x+3}{x^2+9}.$$

The integral is

$$\int \frac{1}{x-2} + \frac{2x+3}{x^2+9}dx = \boxed{\ln|x-2| + \ln(x^2+9) + \arctan\frac{x}{3} + C}.$$

Check. The derivative of the proposed answer is

$$\frac{1}{x-2}+\frac{2x}{x^2+9}+\frac{\frac{1}{3}}{(\frac{x}{3})^2+1}=\frac{1}{x-2}+\frac{2x}{x^2+9}+\frac{9(\frac{1}{3})}{9((\frac{x}{3})^2+1)}$$

$$=\frac{1}{x-2}+\frac{2x}{x^2+9}+\frac{3}{(x^2+9)}=\frac{x^2+9+(2x+3)(x-2)}{x^2+9}=\frac{x^2+9+2x^2-x-6}{x^2+9}$$

$$=\frac{3x^2-x+3}{x^2+9}. \checkmark$$

15. **Find** $\int \sin^4 x \cos^3 x dx$ **. Check your answer.**

Save $\cos x$. Convert the other $\cos^2 x$ into $\sin x$'s. Let $u=\sin x$. It follows that $du=\cos x dx$. The problem equals

$$\int \sin^4 x(1-\sin^2 x)\cos x dx=\int u^4(1-u^2)du=\int (u^4-u^6)du=\tfrac{u^5}{5}-\tfrac{u^7}{7}+C$$

$$=\boxed{\frac{\sin^5 x}{5}-\frac{\sin^7 x}{7}+C}.$$

Check. The derivative of the proposed answer is

$$\sin^4 x\cos x-\sin^6 x\cos x=\sin^4 x\cos x(1-\sin^2 x). \checkmark$$

16. **Find** $\int \frac{dx}{x\sqrt{\ln x}}$ **. Check your answer.**

Answer: Let $u=\ln x$. Then $du=\frac{1}{x}dx$. The integral is equal to

$$\int u^{-1/2}du=2\sqrt{u}+C=\boxed{2\sqrt{\ln x}+C}.$$

Check. The derivative of the proposed answer is

$$2(\tfrac{1}{2})\tfrac{1}{\sqrt{\ln x}}\tfrac{1}{x}\checkmark$$

17. **Find** $\int \sec^6 t dt$ **. Check your answer.**

Answer: Save $\sec^2 t$. Convert the remaining $\sec^4 t$ to $\tan t$'s using $\tan^2 t+1=\sec^2 t$. Let $u=\tan t$. It follows that $du=\sec^2 t dt$. The original problem is equal to

$$\int \left(\tan^2 t+1\right)^2\sec^2 t dt=\int (u^2+1)^2 du=\int (u^4+2u^2+1)du=\frac{u^5}{5}+\frac{2u^3}{3}+u+C$$

$$= \boxed{\frac{\tan^5 t}{5} + \frac{2\tan^3 t}{3} + \tan t + C}.$$

Check . The derivative of the proposed answer is

$$\tan^4 t \sec^2 t + 2\tan^2 t \sec^2 t + \sec^2 t = \sec^2 t(\tan^4 + 2\tan^2 t + 1) = \sec^2 t(\tan^2 t + 1)^2$$

$$= \sec^2 t \sec^4 t. \checkmark$$

18. **Find** $\int \tan^5 x dx$ **. Check your answer.**

Answer: Save $\tan x$. Convert the remaining $\tan^4 x$ to $\sec x$'s using $\tan^2 x + 1 = \sec^2 x$. Let $u = \sec x$. It follows that $du = \sec x \tan x dx$. The original problem is equal to

$$\int \left(\sec^2 x - 1\right)^2 \tan x dx = \int \frac{(u^2-1)^2}{u} du = \int (u^3 - 2u + \tfrac{1}{u}) du$$

$$= \frac{u^4}{4} - 2u^2 + \ln|u| + C = \boxed{\frac{\sec^4 x}{4} - \sec^2 x + \ln|\sec x| + C}.$$

Check . The derivative of the proposed answer is

$$\sec^3 x \sec x \tan x - 2\sec x \sec x \tan x + \tan x = \tan x(\sec^4 x - 2\sec^2 x + 1) =$$

$$\tan x(\sec^2 x - 1)^2 = \tan x \tan^4 x. \checkmark$$

19. **Find** $\int \sqrt{5 + 4x - x^2} dx$ **. Check your answer.**

Answer: Complete the square $5 + 4x - x^2 = 5 + \boxed{4} - (x^2 - 4x + \boxed{4}) = 9 - (x-2)^2$. We let $x - 2 = 3\sin\theta$. It follows that $dx = 3\cos\theta d\theta$ and $9 - (x-2)^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$. The original problem is

$$\int \sqrt{5 + 4x - x^2} dx = \int \sqrt{9 - (x-2)^2} dx = 9\int \cos^2\theta d\theta = \frac{9}{2}\int (1 + \cos 2\theta) d\theta$$

$$= \frac{9}{2}(\theta + (1/2)\sin 2\theta) + C = \frac{9}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\frac{\sqrt{9-(x-2)^2}}{3}\right) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\frac{\sqrt{5+4x-x^2}}{3}\right) + C$$

$$= \boxed{\frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C}$$

Check . The derivative of the proposed answer is

$$\frac{9}{2}\frac{1/3}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}} + (1/2)\left[(x-2)\frac{4-2x}{2\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{9}{2}\frac{1/3}{\frac{1}{3}\sqrt{9-(x-2)^2}} + (1/2)\left[(x-2)\frac{2-x}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{9}{2}\frac{1}{\sqrt{5+4x-x^2}} + (1/2)\left[(x-2)\frac{2-x}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{1}{2\sqrt{5+4x-x^2}}\left[9-(x-2)^2+5+4x-x^2\right]$$

$$= \frac{1}{2\sqrt{5+4x-x^2}}\left[2(5+4x-x^2)\right] = \sqrt{5+4x-x^2}. \checkmark$$

20. **Find** $\int\sqrt{x^2+2x}dx$ **. Check your answer.**

Answer: We complete the square: $x^2+2x = (x^2+2x+1)-1 = (x+1)^2-1$. Let $x+1=\sec\theta$. It follows that $(x+1)^2-1=\tan^2\theta$ and $dx=\sec\theta\tan\theta d\theta$. The original problem is equal to

$$\int \tan^2\theta\sec\theta d\theta.$$

We use integration by parts. Let $u=\tan\theta$ and $dv=\sec\theta\tan\theta d\theta$. It follows that $du=\sec^2\theta d\theta$ and $v=\sec\theta$. So

$$\int\tan^2\theta\sec\theta d\theta = \sec\theta\tan\theta - \int\sec^3\theta d\theta = \sec\theta\tan\theta - \int(\tan^2\theta+1)\sec\theta d\theta$$

$$= \sec\theta\tan\theta - \int\sec\theta d\theta - \int\tan^2\theta\sec\theta d\theta.$$

Add $\int \tan^2\theta \sec\theta d\theta$ to both sides to see that

$$2\int \tan^2\theta\sec\theta d\theta = \sec\theta\tan\theta - \int \sec\theta d\theta.$$

So

$$\int \sqrt{x^2+2x}dx = \int \tan^2\theta\sec\theta d\theta = (1/2)\left[\sec\theta\tan\theta - \int\sec\theta d\theta\right]$$

$$= (1/2)\left[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right] + C$$

$$= \boxed{(1/2)\left[(x+1)\sqrt{x^2+2x} - \ln|(x+1)+\sqrt{x^2+2x}|\right] + C}.$$

<u>Check</u>. The derivative of

$$(1/2)\left[(x+1)\sqrt{x^2+2x} - \ln[(x+1)+\sqrt{x^2+2x}\right]$$

is

$$(1/2)\left[\frac{(x+1)(2x+2)}{2\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1+\frac{2x+2}{2\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1+\frac{x+1}{\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{\sqrt{x^2+2x}+x+1}{[(x+1)+\sqrt{x^2+2x}]\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1}{\sqrt{x^2+2x}}\right]$$

$$= \frac{1}{2\sqrt{x^2+2x}}\left[(x+1)^2 + x^2 + 2x - 1\right]$$

$$= \frac{1}{2\sqrt{x^2+2x}}\left[2x^2+4x\right]$$

$$= \frac{1}{\sqrt{x^2+2x}}\left[x^2+2x\right] = \sqrt{x^2+2x}. \checkmark$$