Math 142, Final Exam , Fall 2006, Solutions

There are 20 problems. Each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Find** $\displaystyle\int \frac{\sin x}{\sqrt{\cos x + 1}}\,dx$ **. Check your answer.**

Let $u = \cos x + 1$. Then $du = -\sin x\,dx$ and the integral is equal to

$$-\int u^{-1/2} du = -2\sqrt{u} + c = \boxed{-2\sqrt{\cos x + 1} + C.}$$

Check: The derivative of the proposed answer is $-2(1/2)(\cos x + 1)^{-1/2}(-\sin x)$. ✓

2. **Find** $\displaystyle\int \sin^4 x \cos^3 x\,dx$ **. Check your answer.**

The integral is equal to

$$\int \sin^4 x(1 - \sin^2 x)\cos x\,dx.$$

Let $u = \sin x$. It follows that $du = \cos x$, and the integral equals

$$\int u^4(1-u^2)\,du = \int (u^4 - u^6)\,du = u^5/5 - u^7/7 + C = \boxed{\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.}$$

Check: The derivative of the proposed answer is

$$\sin^4 x \cos x - \sin^6 x \cos x = \sin^4 x \cos x(1 - \sin^2 x). \checkmark$$

3. **Find** $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$ **. Check your answer.**

The denominator does not factor. Complete the square: $x^2-4x+8 = x^2-4x+4+4 = (x-2)^2+4$. Let $u = x-2$. It follows that $du = dx$ and $x = u+2$. The integral is equal to

$$\int \frac{u+2}{u^2+4}\,du = \int \frac{u}{u^2+4} + \frac{2}{4[(u/2)^2+1]}\,du = 1/2\ln(u^2+4) + \int \frac{4}{4[w^2+1]}\,dw$$

$$= (1/2)\ln(u^2+4) + \arctan w + C = (1/2)\ln(u^2+4) + \arctan(u/2) + C$$

$$= \boxed{(1/2)\ln((x-2)^2+4) + \arctan((x-2)/2) + C},$$

where $w = u/2$ and $dw = du/2$.

Check: The derivative of the proposed answer is

$$\frac{x-2}{x^2-4x+8} + \frac{\frac{1}{2}}{\left(\frac{(x-2)}{2}\right)^2+1} = \frac{x-2}{x^2-4x+8} + \frac{4(\frac{1}{2})}{(x-2)^2+4} \checkmark$$

4. **Find** $\displaystyle\int \arctan x\,dx$ **. Check your answer.**

Use integration by parts. Let $u = \arctan x$ and $dv = dx$. Compute $du = dx/(1+x^2)$ and $v = x$. The original integral is equal to

$$uv - \int v du = x\arctan x - \int \frac{xdx}{1+x^2} = \boxed{x\arctan x - (1/2)\ln(1+x^2) + C.}$$

Check: The derivative of the proposed answer is

$$x\frac{1}{1+x^2} + \arctan x - \frac{x}{1+x^2}. \checkmark$$

5. **Find** $\displaystyle\int \frac{2x+4}{x^3-2x^2}dx$ **. Check your answer.**

Set

$$\frac{2x+4}{x^3-2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

and solve for A, B, and C. Clear the denominator

$$2x+4 = Ax(x-2) + B(x-2) + Cx^2.$$

Plug in zero to see that $4 = -2B$; so, $B = -2$. Plug in 2 to see that $8 = 4C$; so, $C = 2$. Plug in 1 to see that $6 = -A - (-2) + (2)$; so, $A = -2$.

Lets check what we have so far: We see that

$$\frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2} = \frac{-2x(x-2) - 2(x-2) + 2x^2}{x^2(x-2)} = \frac{-2x^2 + 4x - 2x + 4 + 2x^2}{x^2(x-2)}$$

$$= \frac{2x+4}{x^3 - 2x^2}. \checkmark$$

So the original integral is

$$\int \left(\frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}\right) dx = \boxed{-2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C}.$$

6. **Find** $\lim\limits_{x\to\infty} \left(\dfrac{x}{x+5}\right)^x$ **.**

This limit is equal to

$$\lim_{x\to\infty} \left(\frac{x+5-5}{x+5}\right)^x = \lim_{x\to\infty} \left(1 + \frac{-5}{x+5}\right)^x.$$

Let $t = x + 5$. The limit is

$$\lim_{t\to\infty} \left(1 + \frac{-5}{t}\right)^{t-5} = \lim_{t\to\infty} \left(1 + \frac{-5}{t}\right)^t \left(1 + \frac{-5}{t}\right)^{-5}.$$

I know that $\lim\limits_{t\to\infty} \left(1 + \frac{r}{t}\right)^t = e^r$; so, the answer to our question is

$$e^{-5}(1)^{-5} = \boxed{e^{-5}}.$$

7. **Find the area between** $y^2 = x$ **and** $y = x - 2$ **.**

I drew a picture elsewhere. The graphs intersect at $(4, 2)$ and $(1, -1)$. I chop the y-axis from $y = -1$ to $y = 2$. I integrate the big x minus the little x. The area is

$$\int_{-1}^{2} (y + 2 - y^2)dy = y^2/2 + 2y - y^3/3\Big|_{-1}^{2} = \boxed{2 + 4 - 8/3 - (1/2 - 2 + 1/3).}$$

8. **Consider the sequence $\{a_n\}$ with $a_1 = \sqrt{20}$, and $a_n = \sqrt{20 + a_{n-1}}$ for $n \ge 20$. Prove that the sequence $\{a_n\}$ converges. Find the limit of the sequence $\{a_n\}$.**

Notice that $a_n \le 5$ for all n. It is clear that $a_1 < 5$. If $a_{n-1} \le 5$, then $a_{n-1} + 20 \le 25$; hence, $a_n = \sqrt{a_{n-1} + 20} \le \sqrt{25} = 5$. Our assertion is established by Mathematical Induction.

We now claim that the sequence $\{a_n\}$ is an increasing sequence. We know that $a_{n-1} \le 5$. Multiply both sides by the positive number $a_{n-1} + 4$ to see that $a_{n-1}^2 + 4a_{n-1} \le 5a_{n-1} + 20$. In other words, $a_{n-1}^2 \le a_{n-1} + 20$. The square root function is an increasing function; so, $a_{n-1} \le \sqrt{a_{n-1} + 20} = a_n$. Our claim is established.

The sequence $\{a_n\}$ is an increasing sequence which never gets beyond 5. The Completeness axiom guarantees that the sequence converges. Let L be the name of $\lim_{n \to \infty} a_n$. Take the limit of $a_n = \sqrt{20 + a_{n-1}}$ to see that $L = \sqrt{L + 20}$. We can now solve for L. We have $L^2 = L + 20$ or $L^2 - L - 20 = 0$. We factor to get $(L-5)(L+4) = 0$. So, $L = -4$ or $L = 5$. All of our a_n are positive so L can not possibly negative. We conclude that $\boxed{L = 5}$.

9. **A conical water tank sits with its base on the ground. The radius of the base is 10 feet. The height of the tank is 30 feet. The tank is filled to a depth of 25 feet. How much work is required to pump all of the water out through a hole in the top of the tank? The density of water is 62.4 lb/ft ^{3}. Be sure to give the units for your answer.**

I drew a picture elsewhere. Notice that I arranged my axis, so that $x = 0$ is the top of the tank. The water starts at $x = 5$. The bottom of the water occurs at $x = 30$. For each x between 5 and 30, we lift a thin layer of water starting at x-coordinate x. The work to lift this thin layer is the weight of the layer times the distance this layer must be lifted. The distance is x. (That is the advantage of the way I set my axis.) The weight of the layer is the volume of the layer times the density of water. The volume of the layer is the area of the top times the thickness. The thickness is dx and the area of the top is πr^2, where similar triangles tell us that $r = \frac{1}{3}x$. The work to lift the layer of water at x-coordinate x is

$$(62.4)\pi(\tfrac{1}{3}x)^2 x dx.$$

The total work is

$$\frac{(62.4)\pi}{9}\int_5^{30} x^3 dx = \frac{(62.4)\pi}{9}\frac{x^4}{4}\Big|_5^{30} = \boxed{\frac{(62.4)\pi}{36}[30^4 - 5^4] \text{ foot-pounds.}}$$

10. **Consider the region in the first quadrant which is bounded by $y = x^2$, the x-axis, and $x = 1$. Revolve this region about the line $y = 5$. What is the volume of the resulting solid?**

I drew a picture elsewhere. I chop the y-axis from $y = 0$ to $y = 1$. For each little piece of the y-axis I draw a rectangle from $x = \sqrt{y}$ to $x = 1$. I spin the rectangle and get a shell of volume $2\pi rht$, where $t = dy$, $h = 1 - \sqrt{y}$, and $r = 5 - y$. The volume of the solid is

$$2\pi \int_0^1 (1 - \sqrt{y})(5 - y)dy = 2\pi \int_0^1 (y^{3/2} - 5\sqrt{y} - y + 5)dy$$

$$= 2\pi \left((2/5)y^{5/2} - 5(2/3)y^{3/2} - y^2/2 + 5y\right)\Big|_0^1 = \boxed{2\pi((2/5) - 5(2/3) - 1/2 + 5)}$$

11. **Find the length of $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$.**

The length is equal to

$$\int_1^2 \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2} dx = \int_1^2 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2} dx = \int_1^2 \sqrt{1 + \tfrac{9}{4}x} dx = \frac{4}{9}\frac{2}{3}(1 + \tfrac{9}{4}x)^{3/2}\Big|_1^2$$

$$= \boxed{\tfrac{4}{9}\tfrac{2}{3}\left[(1 + \tfrac{9}{4}(2))^{3/2} - (1 + \tfrac{9}{4})^{3/2}\right]}.$$

12. **Let $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(x-3)^k}{6^k k}$. Find all real numbers x for which $f(x)$ converges. Justify your answer.**

We use the ratio test. Let

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|x-3|^{k+1}}{6^{k+1}(k+1)} \frac{6^k k}{|x-3|^k} = \lim_{k\to\infty} \frac{|x-3|}{6} \frac{k}{k+1} = \frac{|x-3|}{6}.$$

If $\rho < 1$, then the series converges. If $1 < \rho$, then the series diverges. We see that $\rho < 1$ precisely when $-3 < x < 9$. We also see that $1 < \rho$ precisely when $x < -3$ or $9 < x$. We need only worry about $x = -3$ and $x = 9$.

We see that

$$f(9) = \sum_{k=1}^{\infty} \frac{(9-3)^k}{6^k k} = \sum_{k=1}^{\infty} \frac{1}{k},$$

which is the harmonic series and diverges. We see that

$$f(-3) = \sum_{k=1}^{\infty} \frac{(-6)^k}{6^k k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k},$$

which is minus the alternating harmonic series and converges. We conclude that

$$\boxed{f(x) \text{ converges for } -3 \le x < 9 \text{ and } f(x) \text{ diverges for all other } x.}$$

13. **Find** $\lim\limits_{x\to 0} \dfrac{e^{x^2}-1-x^2-\frac{x^4}{2}-\frac{x^6}{3!}}{x^8}$ **. Justify your answer.**

We know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

It follows that

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots;$$

and therefore,

$$\lim_{x\to 0} \frac{e^{x^2}-1-x^2-\frac{x^4}{2}-\frac{x^6}{3!}}{x^8}$$

$$= \lim_{x\to 0} \frac{\left(1+x^2+\frac{x^4}{2}+\frac{x^6}{3!}+\frac{x^8}{4!}+\dots\right)-1-x^2-\frac{x^4}{2}-\frac{x^6}{3!}}{x^8}$$

$$= \lim_{x\to 0} \frac{\left(\frac{x^8}{4!}+\dots\right)}{x^8}$$

$$= \lim_{x\to 0} \frac{x^8\left(\frac{1}{4!}+\frac{x^2}{5!}+\frac{x^4}{6!}+\dots\right)}{x^8}$$

$$= \lim_{x\to 0} \left(\frac{1}{4!}+\frac{x^2}{5!}+\frac{x^4}{6!}+\dots\right) = \boxed{\frac{1}{24}.}$$

14. **Does** $\sum\limits_{k=1}^{\infty} \dfrac{k}{k^2+1}$ **converge? Justify your answer.**

Apply the limit comparison test with the divergent harmonic series $\sum\limits_{k=1}^{\infty}\frac{1}{k}$. We see that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2}{k^2+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k^2}} = 1,$$

and 1 is a number, not 0, not ∞. We conclude that both series converge or both series diverge. The series $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ diverges; hence, the series $\sum\limits_{k=1}^{\infty}\frac{k}{k^2+1}$ also DIVERGES.

15. **Does** $\displaystyle\sum_{k=1}^{\infty}\frac{5^k+k}{k!+3}$ **converge? Justify your answer.**

Notice that
$$\frac{5^k+k}{k!+3}<\frac{5^k+5^k}{k!}=\frac{2\cdot 5^k}{k!}$$
because the fraction on the right has a larger numerator and a smaller denominator. The series $\sum\limits_{k=1}^{\infty}\frac{2\cdot 5^k}{k!}$ converges by the ratio test:
$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\frac{\frac{2\cdot 5^{k+1}}{(k+1)!}}{\frac{2\cdot 5^k}{k!}}=\lim_{k\to\infty}\frac{2\cdot 5^{k+1}}{(k+1)!}\frac{k!}{2\cdot 5^k}=\lim_{k\to\infty}\frac{5}{k+1}=0<1.$$
Both series $\sum\limits_{k=1}^{\infty}\frac{5^k+k}{k!+3}$ and $\sum\limits_{k=1}^{\infty}\frac{2\cdot 5^k}{k!}$ are positive series. The terms of $\sum\limits_{k=1}^{\infty}\frac{5^k+k}{k!+3}$ are smaller than the terms of $\sum\limits_{k=1}^{\infty}\frac{2\cdot 5^k}{k!}$. The series $\sum\limits_{k=1}^{\infty}\frac{2\cdot 5^k}{k!}$ converges. We apply the Comparison test to conclude that the series $\sum\limits_{k=1}^{\infty}\frac{5^k+k}{k!+3}$ also converges.

16. **What is the exact sum of the series** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(4^k)}$ **? Justify your answer.**

We know that if $-1<x<1$, then $\frac{1}{1-x}=\sum\limits_{\ell=0}^{\infty}x^\ell$. Integrate to learn that if $-1<x<1$, then $-\ln(1-x)=\sum\limits_{\ell=0}^{\infty}\frac{x^{\ell+1}}{\ell+1}+C$, for some constant C. Plug in $x=0$ to see that $0=0+C$, so $C=0$ and $-\ln(1-x)=\sum\limits_{\ell=0}^{\infty}\frac{x^{\ell+1}}{\ell+1}$, for $-1<x<1$. Let $k=\ell+1$. Notice that when ℓ is 0, then $k=1$. Thus, if $-1<x<1$, then $-\ln(1-x)=\sum\limits_{k=1}^{\infty}\frac{x^k}{k}$. Plug in $x=\frac{1}{4}$ to see that
$$\sum_{k=1}^{\infty}\frac{1}{k(4^k)}=-\ln(1-\tfrac{1}{4})=-\ln(\tfrac{3}{4})=\boxed{\ln(\tfrac{4}{3}).}$$

17. **Approximate** $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^5}$ **with an error at most** $\frac{1}{1000}$ **. Justify your answer.**

We see that
$$\left|\sum_{k=1}^{\infty}\frac{1}{k^5}-\sum_{k=1}^{n}\frac{1}{k^5}\right|=\sum_{k=n+1}^{\infty}\frac{1}{k^5}.$$

I drew some boxes elsewhere to help approximate the right most sum. The sum is the area inside the boxes, which is less than the area under the curve, which equals

$$\int_n^\infty \frac{1}{x^5}dx = \lim_{b\to\infty} \frac{1}{-4x^4}\bigg|_n^b = \lim_{b\to\infty} \frac{1}{-4b^4} + \frac{1}{4n^4} = \frac{1}{4n^4}.$$

We have shown that

$$\left|\sum_{k=1}^{\infty} \frac{1}{k^5} - \sum_{k=1}^{n} \frac{1}{k^5}\right| \le \frac{1}{4n^4}$$

Notice that when $n = 4$ (or higher) $\frac{1}{4n^4} < \frac{1}{1000}$. We conclude that

$$\sum_{k=1}^{4} \frac{1}{k^5}$$

approximates $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5}$ with an error at most $\frac{1}{1000}$.

18. **Approximate $\displaystyle\int_0^{\frac{1}{10}} \sin(x^2)\,dx$ with an error at most $\frac{1}{1000}$. Justify your answer.**

The given integral is equal to

$$\int_0^{\frac{1}{10}} x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\, dx$$

$$= \left(\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \dots\right)\bigg|_0^{\frac{1}{10}}$$

$$= \frac{1}{3\cdot 10^3} - \frac{1}{7\cdot 3!\cdot 10^7} + \frac{1}{11\cdot 5!\cdot 10^{11}} - \frac{1}{15\cdot 7!\cdot 10^{15}} + \dots$$

We have found a series which converges to $\displaystyle\int_0^{\frac{1}{10}} \sin(x^2)\,dx$. We may apply the alternating series test to the series. The series alternates. The (absolute value of the) terms decrease. The terms go to zero. The distance between the sum of the entire series and some particular partial sum is less than the next term in the series. We see that $\frac{1}{7\cdot 3!\cdot 10^7} < \frac{1}{1000}$. We conclude that $\frac{1}{3\cdot 10^3}$ approximates the value of $\displaystyle\int_0^{\frac{1}{10}} \sin(x^2)\,dx$ with an error at most $\frac{1}{1000}$.

19. **Find the Taylor polynomial $P_3(x)$ of order 3 for the function $f(x) = \ln(x+1)$ about $a = 0$.**

We see that

$$f(x) = \ln(x+1), \quad f'(x) = \frac{1}{x+1}, \quad f''(x) = \frac{-1}{(x+1)^2}, \quad f'''(x) = \frac{2}{(x+1)^3},$$

$$f^{(4)}(x) = \frac{-6}{(x+1)^4},$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad \text{and} \quad f'''(0) = 2.$$

We know that $P_3(x) = f(0) + f'(1)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x{-}^3$. Thus,

$$\boxed{P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.}$$

20. **Keep the notation of problem 19. Find an upper bound for the error that is introduced if $P_3(x)$ is used to approximate $f(x)$ when $|x| < .1$**

We know that

$$|f(x) - P_3(x)| = |R_3(x)| = \left|\frac{f^{(4)}(c)x^4}{4!}\right| = \left|\frac{-6x^4}{(c+1)^4 4!}\right| = \frac{|x|^4}{|c+1|^4 4},$$

for some c between x and 0. We are told that $|x| < .1$. So

$$|R_3(x)| \le \frac{1}{|c+1|^4 4(10)^4}.$$

We know that $-.1 < x < .1$ and c is between x and 0. So, $-.1 < c < .1$ and $1 - .1 < 1 + c < 1 + .1$. In other words, $.9 < c + 1$. It follows that $\frac{1}{c+1} < \frac{10}{9}$, and

$$|R_3(x)| \le \frac{10^4}{9^4 4(10)^4} = \frac{1}{9^4 4}.$$

We conclude that: if $P_3(x)$ is used to approximate $f(x)$ when $|x| < .1$, then the error that is introduced is less than $\frac{1}{9^4 4}$.