Math 142, Final Exam , Spring 2006

There are 20 problems. Each problem is worth 10 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Find** $\int \frac{e^x}{\sqrt{e^x+1}}\,dx$ **. Check your answer.**

$$\boxed{2\sqrt{e^x+1}+C.}$$

The derivative of the proposed answer is

$$2(\tfrac{1}{2})\tfrac{e^x}{\sqrt{e^x+1}}. \checkmark$$

2. **Find** $\int \sin^3 x\cos^2 x\,dx$ **. Check your answer.**

The integral is equal to

$$\int \sin x(1-\cos^2 x)\cos^2 x\,dx = \int \sin x(\cos^2 x-\cos^4 x)\,dx.$$

Let $u=\cos x$. It follows that $du=-\sin x\,dx$, and the integral is equal to

$$-\int (u^2-u^4)\,du = \boxed{-\tfrac{\cos^3 x}{3}+\tfrac{\cos^5 x}{5}+C.}$$

The derivative of the proposed answer is

$$-\cos^2 x(-\sin x)+\cos^4 x(-\sin x) = \cos^2 x\sin x(1-\cos^2 x). \checkmark$$

3. **Find** $\int \sqrt{x^2+1}\,dx$ **. Check your answer.**

Let $x=\tan\theta$. It follows that $dx=\sec^2\theta d\theta$ and $\sqrt{x^2+1}=\sec\theta$. So the integral is equal to

$$\int \sec^3\theta d\theta.$$

Let $u = \sec\theta$ and $dv = \sec^2\theta d\theta$. It follows that $du = \sec\theta\tan\theta d\theta$ and $v = \tan\theta$. Apply integration by parts to see that

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta.$$

In other words,

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta.$$

or

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta.$$

It follows that

$$\textstyle\int \sec^3\theta d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$

$$= \boxed{\tfrac{1}{2}\left(x\sqrt{x^2+1} + \ln|\sqrt{x^2+1}+x|\right) + C.}$$

The derivative of the proposed answer is

$$\tfrac{1}{2}\left(x\tfrac{x}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \tfrac{\frac{x}{\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x}\right)$$

$$= \tfrac{1}{2}\left(x\tfrac{x}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \tfrac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}(\sqrt{x^2+1}+x)}\right)$$

$$= \tfrac{1}{2}\left(x\tfrac{x}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \tfrac{1}{\sqrt{x^2+1}}\right)$$

$$= \tfrac{1}{2}\left(\tfrac{x^2+1}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right). \checkmark$$

4. **Find** $\displaystyle\int \ln x\,dx$ **. Check your answer.**

Let $u = \ln x$ and $dv = dx$. It follows that $du = \frac{dx}{x}$ and $v = x$. Use integration by parts to see that the integral is equal to

$$x\ln x - \int dx = \boxed{x\ln x - x + C.}$$

The derivative of the proposed answer is

$$1 + \ln x - 1. \checkmark$$

5. **Find** $\displaystyle\int \frac{4x^3+6x^2+x+2}{x^4+x^2}dx$ **. Check your answer.**

A quick calculation shows that

$$\int \frac{4x^3+6x^2+x+2}{x^4+x^2}dx = \int \frac{1}{x}+\frac{2}{x^2}+\frac{3x+4}{x^2+1}dx$$

$$= \boxed{\ln|x| + \frac{-2}{x} + \tfrac{3}{2}\ln(x^2+1) + 4\arctan x + C.}$$

The derivative of the proposed answer is

$$\frac{1}{x}+\frac{2}{x^2}+\frac{3x}{x^2+1}+\frac{4}{1+x^2}$$

$$= \frac{(x^3+x)+2(x^2+1)+3x^3+4x^2}{x^2+x^4}$$

$$= \frac{4x^3+6x^2+x+2}{x^2+x^4}. \checkmark$$

6. **Find** $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x-4}\right)^x$ **.**

The limit is equal to

$$\lim_{x\to\infty}\left(\frac{(x-4)+4}{x-4}\right)^x = \lim_{x\to\infty}\left(1+\frac{4}{x-4}\right)^x.$$

Let $t = x-4$. The limit is equal to

$$= \lim_{t\to\infty}\left(1+\frac{4}{t}\right)^{t+4} = \lim_{t\to\infty}\left(1+\frac{4}{t}\right)^{t}\lim_{t\to\infty}\left(1+\frac{4}{t}\right)^{4}.$$

We know that $\lim_{t\to\infty}(1+\frac{r}{t})^t = e^r$. We conclude that the answer is $\boxed{e^4}$.

7. **Find the area between** $y = x$ **and** $x + y^2 = 2$ **.**

I drew a picture elsewhere. The intersection points are $(1,1)$ and $(-2,-2)$. We partition the y-axis. The area is

$$\int_{-2}^{1}[(2-y^2)-y]\,dy = 2y - \frac{y^3}{3} - \frac{y^2}{2}\bigg|_{-2}^{1} = \boxed{2-\frac{1}{3}-\frac{1}{2}-(-4+\frac{8}{3}-2)} = \frac{27}{6}$$

8. **Consider the sequence $\{a_n\}$ with $a_1 = 10$, and $a_n = \frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}]$ for $n \geq 2$. Prove that the sequence $\{a_n\}$ converges. Find the limit of the sequence $\{a_n\}$.**

Suppose, for the time being, that the sequence converges. Let $L = \lim_{n\to\infty} a_n$. Take the limit of both sides of $a_n = \frac{1}{2}[a_{n-1} + \frac{3}{a_{n-1}}]$ to see that

$$L = \tfrac{1}{2}[L + \tfrac{3}{L}].$$

Multiply both sides by $2L$ to see that $2L^2 = L^2+3$; so, $L^2 = 3$ and L is equal to $\sqrt{3}$ or $-\sqrt{3}$. All of the numbers a_n are non-negative; so L most be non-negative. We now know that, if L exists, then L must be $\sqrt{3}$.

We still have to prove that L exists. I will show that the sequence $\{a_n\}$ is a decreasing sequence of Real numbers which is bounded below by $\sqrt{3}$. The (dual of the) Completeness axiom tells us that the sequence $\{a_n\}$ has a limit.

I first show that $\sqrt{3} \leq a_n$ for all n. We see that $3 \leq a_1$. In general, we hope to show that

$$\sqrt{3} \leq \tfrac{1}{2}[a_{n-1} + \tfrac{3}{a_{n-1}}].$$

Multiply both sides by the positive number $2a_{n-1}$. We hope to show

$$2\sqrt{3}a_{n-1} \leq a_{n-1}^2 + 3.$$

We hope to show that

$$0 \leq a_{n-1}^2 - 2\sqrt{3}a_{n-1} + 3.$$

The right side factors as $(a_{n-1} - \sqrt{3})^2$, and this perfect square is non-negative. Read the calculation from the bottom up to see that $\sqrt{3} \leq a_n$ for all n.

Finally, I show that $a_n \leq a_{n-1}$, for all $n \geq 2$. I will show that

$$\tfrac{1}{2}[a_{n-1} + \tfrac{3}{a_{n-1}}] \leq a_{n-1}.$$

Multiply by the positive number $2a_{n-1}$. We hope to show that

$$a_{n-1}^2 + 3 \leq 2a_{n-1}^2.$$

We hope to show that

$$0 \leq a_{n-1}^2 - 3$$

We hope to show that

$$0 \leq (a_{n-1} + \sqrt{3})(a_{n-1} - \sqrt{3}).$$

Divide by the positive number $(a_{n-1} + \sqrt{3})$. We hope to show

$$0 \leq a_{n-1} - \sqrt{3}.$$

Fortunately, we have already shown that every member of the sequence is at least $\sqrt{3}$. Read the calculation from the bottom to the top to see that $a_n \leq a_{n-1}$.

9. **A conical water tank sits with its base on the ground. The radius of the base is 10 feet. The height of the tank is 30 feet. The tank is filled to a depth of 25 feet. How much work is required to pump all of the water out through a hole in the top of the tank? The density of water is 62.4 lb/ft ^{3}. Be sure to give the units for your answer.**

I drew a picture elsewhere. Notice that I arranged my axis, so that $x=0$ is the top of the tank. The water starts at $x=5$. The bottom of the water occurs at $x=30$. For each x between 5 and 30, we lift a thin layer of water starting at x-coordinate x. The work to lift this thin layer is the weight of the layer times the distance this layer must be lifted. The distance is x. (That is the advantage of the way I set my axis.) The weight of the layer is the volume of the layer times the density of water. The volume of the layer is the area of the top times the thickness. The thickness is dx and the area of the top is πr^2, where similar triangles tell us that $r=\frac{1}{3}x$. The work to lift the layer of water at x-coordinate x is

$$(62.4)\pi(\tfrac{1}{3}x)^2 x dx.$$

The total work is

$$\frac{(62.4)\pi}{9}\int_5^{30} x^3 dx = \frac{(62.4)\pi}{9}\frac{x^4}{4}\bigg|_5^{30} = \boxed{\frac{(62.4)\pi}{36}[30^4-5^4] \text{ foot-pounds.}}$$

10. **Consider the region in the first quadrant which is bounded by $y=x^2$, the x-axis, and $x=1$. Revolve this region about the line $x=5$. What is the volume of the resulting solid?**

I have drawn a picture on a different page. I partition the x-axis, and draw rectangles which are perpendicular to the x-axis. Spin each rectangle and get a cylindrical shell of volume $2\pi rht$, where $t=dx$, $r=5-x$, and $h=x^2$. The volume is

$$2\pi\int_0^1 (5-x)x^2\,dx = 2\pi\int_0^1 (5x^2-x^3)\,dx = 2\pi\left[\frac{5x^3}{3}-\frac{x^4}{4}\right]_0^1 = \boxed{2\pi\left(\frac{5}{3}-\frac{1}{4}\right)} = \frac{17\pi}{6}.$$

11. **Find the length of $24xy=y^4+48$ from $y=2$ to $y=4$.**

The equation is $x=\frac{y^3}{24}+\frac{2}{y}$. The length is

$$\int_2^4\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy = \int_2^4\sqrt{1+\left(\frac{y^2}{8}-\frac{2}{y^2}\right)^2}dy = \int_2^4\sqrt{1+\frac{y^4}{64}-\frac{1}{2}+\frac{4}{y^4}}dy$$

$$=\int_2^4\sqrt{\frac{y^4}{64}+\frac{1}{2}+\frac{4}{y^4}}dy = \int_2^4\sqrt{\left(\frac{y^2}{8}+\frac{2}{y^2}\right)^2}dy = \int_2^4\left(\frac{y^2}{8}+\frac{2}{y^2}\right)dy = \frac{y^3}{24}-\frac{2}{y}\bigg|_2^4$$

$$=\boxed{\tfrac{4^3}{24}-\tfrac{1}{2}-(\tfrac{2^3}{24}-1)} = \tfrac{17}{6}$$

12. **Let** $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{5^k k}$ **. Find all real numbers** x **for which** $f(x)$ **converges. Justify your answer.**

We use the ratio test. Let

$$\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|x-2|^{k+1}}{5^{k+1}(k+1)} \frac{5^k k}{|x-2|^k} = \lim_{k\to\infty} \frac{|x-2|}{5} \frac{k}{k+1} = \frac{|x-2|}{5}.$$

If $\rho < 1$, then the series converges. If $1 < \rho$, then the series diverges. We see that $\rho < 1$ precisely when $-3 < x < 7$. We also see that $1 < \rho$ precisely when $x < -3$ or $7 < x$. We need only worry about $x = -3$ and $x = 7$.

We see that

$$f(7) = \sum_{k=1}^{\infty} \frac{(7-2)^k}{5^k k} = \sum_{k=1}^{\infty} \frac{1}{k},$$

which is the harmonic series and diverges. We see that

$$f(-3) = \sum_{k=1}^{\infty} \frac{(-5)^k}{5^k k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k},$$

which is minus the alternating harmonic series and converges. We conclude that

$$\boxed{f(x) \text{ converges for } -3 \le x < 7 \text{ and } f(x) \text{ diverges for all other } x.}$$

13. **Find** $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{3!}}{x^8}$ **. Justify your answer.**

We know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

It follows that

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots;$$

and therefore,

$$\begin{aligned}
&\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{3!}}{x^8} \\
&= \lim_{x\to 0} \frac{\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots\right) - 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{3!}}{x^8} \\
&= \lim_{x\to 0} \frac{\left(\frac{x^8}{4!} + \dots\right)}{x^8} \\
&= \lim_{x\to 0} \frac{x^8\left(\frac{1}{4!} + \frac{x^2}{5!} + \frac{x^4}{6!} + \dots\right)}{x^8} \\
&= \lim_{x\to 0} \left(\frac{1}{4!} + \frac{x^2}{5!} + \frac{x^4}{6!} + \dots\right) = \boxed{\frac{1}{24}.}
\end{aligned}$$

14. **Does** $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$ **converge? Justify your answer.**

The series $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ is the p-series with $p = \frac{5}{2} > 1$; so $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ converges. I apply the limit comparison test to the two series considered so far. I see that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\sqrt{k}}{k^3+1}}{\frac{1}{k^{5/2}}} = \lim_{k\to\infty} \frac{k^{5/2}\sqrt{k}}{k^3+1} = \lim_{k\to\infty} \frac{k^3}{k^3+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k^3}} = 1.$$

This limit is a number, not 0, not ∞. The Limit Comparison Test tells us that both series converge or both series diverge. We have seen that $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ converges. We conclude that

$$\boxed{\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1} \text{ converges.}}$$

15. **Does** $\sum_{k=1}^{\infty} \frac{5^k+k}{k!+3}$ **converge? Justify your answer.**

Notice that

$$\frac{5^k+k}{k!+3} < \frac{5^k+5^k}{k!} = \frac{2\cdot 5^k}{k!}$$

because the fraction on the right has a larger numerator and a smaller denominator. The series $\sum_{k=1}^{\infty} \frac{2\cdot 5^k}{k!}$ converges by the ratio test:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{2\cdot 5^{k+1}}{(k+1)!}}{\frac{2\cdot 5^k}{k!}} = \lim_{k\to\infty} \frac{2\cdot 5^{k+1}}{(k+1)!}\frac{k!}{2\cdot 5^k} = \lim_{k\to\infty} \frac{5}{k+1} = 0 < 1.$$

Both series $\sum_{k=1}^{\infty} \frac{5^k+k}{k!+3}$ and $\sum_{k=1}^{\infty} \frac{2\cdot 5^k}{k!}$ are positive series. The terms of $\sum_{k=1}^{\infty} \frac{5^k+k}{k!+3}$ are smaller than the terms of $\sum_{k=1}^{\infty} \frac{2\cdot 5^k}{k!}$. The series $\sum_{k=1}^{\infty} \frac{2\cdot 5^k}{k!}$ converges. We apply the Comparison test to conclude that the series $\sum_{k=1}^{\infty} \frac{5^k+k}{k!+3}$ also converges.

16. **What is the exact sum of the series** $\sum_{k=0}^{\infty} \frac{k}{3^k}$ **? Justify your answer.**

We know that $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for all real numbers x with $-1 < x < 1$. Take the derivative of both sides to see that

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

for all real numbers x with $-1 < x < 1$. Multiply both sides by x to see

$$\sum_{k=0}^{\infty} kx^{k} = \frac{x}{(1-x)^2}$$

for all real numbers x with $-1 < x < 1$. Plug in $x = \frac{1}{3}$ to conclude that

$$\sum_{k=0}^{\infty} \frac{k}{3^k} = \boxed{\frac{\frac{1}{3}}{(1-\frac{1}{3})^2}.}$$

17. **Approximate** $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ **with an error at most** $\frac{1}{1000}$ **. Justify your answer.**

We see that

$$\left|\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{n} \frac{1}{k^4}\right| = \sum_{k=n+1}^{\infty} \frac{1}{k^4}.$$

I drew some boxes elsewhere to help approximate the right most sum. The sum is the area inside the boxes, which is less than the area under the curve, which equals

$$\int_n^{\infty} \frac{1}{x^4} dx = \lim_{b\to\infty} \frac{1}{-3x^3}\bigg|_n^b = \lim_{b\to\infty} \frac{1}{-3b^3} + \frac{1}{3n^3} = \frac{1}{3n^3}.$$

We have shown that

$$\left|\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{n} \frac{1}{k^4}\right| \le \frac{1}{3n^3}$$

Notice that when $n = 7$ (or higher) $\frac{1}{3n^3} < \frac{1}{1000}$. We conclude that

$$\sum_{k=1}^{7} \frac{1}{k^4}$$

approximates $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ with an error at most $\frac{1}{1000}$.

18. **Approximate** $\int_0^{\frac{1}{10}} \sin(x^2)\,dx$ **with an error at most** $\frac{1}{1000}$ **. Justify your answer.**

The given integral is equal to

$$\int_0^{\frac{1}{10}} x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\, dx$$

$$= \left(\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \dots\right)\Big|_0^{\frac{1}{10}}$$

$$= \frac{1}{3\cdot 10^3} - \frac{1}{7\cdot 3!\cdot 10^7} + \frac{1}{11\cdot 5!\cdot 10^{11}} - \frac{1}{15\cdot 7!\cdot 10^{15}} + \dots$$

We have found a series which converges to $\int_0^{\frac{1}{10}} \sin(x^2)\,dx$. We may apply the alternating series test to the series. The series alternates. The (absolute value of the) terms decrease. The terms go to zero. The distance between the sum of the entire series and some particular partial sum is less than the next term in the series. We see that $\frac{1}{7\cdot 3!\cdot 10^7} < \frac{1}{1000}$. We conclude that $\frac{1}{3\cdot 10^3}$ approximates the value of $\int_0^{\frac{1}{10}} \sin(x^2)\,dx$ with an error at most $\frac{1}{1000}$.

19. **Find the Taylor polynomial** $P_3(x)$ **of order** 3 **for the function** $f(x) = \ln x$ **about** $a = 1$**.**

We see that

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = \frac{-1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = \frac{-6}{x^4},$$

$$f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad \text{and} \quad f'''(x) = 2.$$

We know that $P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$. Thus,

$$\boxed{P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.}$$

20. **Keep the notation of problem 19. Find an upper bound for the error that is introduced if** $P_3(x)$ **is used to approximate** $f(x)$ **when** $|x-1| < .1$

We know that

$$|f(x) - P_3(x)| = |R_3(x)| = \left|\frac{f^{(4)}(c)(x-1)^4}{4!}\right| = \left|\frac{-6(x-1)^4}{c^4 4!}\right| = \frac{|x-1|^4}{|c|^4 4},$$

for some c between x and 1. We are told that $|x-1|<.1$. So

$$|R_3(x)| \le \frac{1}{|c|^4 4(10)^4}.$$

We know that $.9<x<1.1$; so $.9<c$. It follows that $\frac{1}{c}<\frac{10}{9}$, and

$$|R_3(x)| \le \frac{10^4}{9^4 4(10)^4} = \frac{1}{9^4 4}.$$

We conclude that: if $P_3(x)$ is used to approximate $f(x)$ when $|x-1|<.1$, then the error that is introduced is less than $\frac{1}{9^4 4}$.