Math 142, Exam 3, Fall 2006

Write your answers as legibly as you can on the blank sheets of paper provided.

Please leave room in the upper left corner for the staple.

Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 100 points. There are 10 problems. Each problem is worth 10 points.

SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators or Cell phones.**

I will post the solutions on my website sometime this afternoon.

If I know your e-mail address, I will e-mail your grade to you as soon as the exam is graded. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

1. **A cone shaped water reservoir is 20 feet in diameter across the top and 15 feet deep. If the reservoir is filled to a depth of 10 feet, how much work is required to pump all the water to the top of the reservoir? The density of water is $62.4\,\text{lb/ft}^3$. (There is no need for you to do arithmetic. Leave your answer in terms of sums and products of numbers.)**

Look at the picture. Notice that my axis is aimed downward, with $y=0$ at the top. The RADIUS of the top is 10. The radius of the thin layer of water whose y-coordinate is y is $10-\frac{2}{3}y$. Draw similar triangles or merely find the equation of the line that includes the points $r=10$, $y=0$ and $r=0$, $y=15$. The work to lift the layer with y-cordinate y is the weight of the layer times the distance this layer is to be lifted, and this is

$$\text{vol} \times \text{density} \times \text{distance}$$

$$= \pi r^2 t(62.4)y = \pi\left(10-\frac{2}{3}y\right)^2(62.4)y\,dy,$$

with $t = dy$. At the beginning the water water covers from $y = 5$ to $y = 15$. The work required to pump all of the water out of the tank is

$$\int_5^{15} \pi \left(10 - \frac{2}{3}y\right)^2 y(62.4))dy$$

$$= (62.4)\pi \int_5^{15} \left(100y - \frac{40}{3}y^2 + \frac{4}{9}y^3\right) dy$$

$$\boxed{= (62.4)\pi \left(50y^2 - \frac{40}{9}y^3 + \frac{1}{9}y^4\right)\Big|_5^{15} \text{ foot-pounds.}}$$

2. **Find** $\displaystyle\int_0^5 \frac{1}{(x-4)^2}\,dx$ **.**

Draw the picture. Notice that the graph is always above the x-axis and that the graph goes to ∞ as x approaches 4. Hence the answer is either some positive number or else the integral diverges to $+\infty$. The integral is equal to

$$\lim_{b\to 4^-} \int_0^b \frac{1}{(x-4)^2}dx + \lim_{a\to 4^+} \int_a^5 \frac{1}{(x-4)^2}dx$$

$$= \lim_{b\to 4^-} \frac{-1}{(x-4)}\Big|_0^b + \lim_{a\to a^+} \frac{-1}{x-4}\Big|_a^5$$

$$= \lim_{b\to 4^-} \left(\frac{-1}{b-4} - \frac{1}{4}\right) + \lim_{a\to 4^+} \left(-1 + \frac{1}{a-4}\right).$$

Both limits are $+\infty$. $\boxed{\text{The integral diverges to } +\infty.}$ (The number $-\frac{5}{4}$ has nothing to do with this problem. If your answer is $-\frac{5}{4}$, then your score on this problem will be 0.)

3. **Find** $\displaystyle\int \sin^3 x\,dx$. **Check your answer.**

THe integral is $\int(1-\cos^2 x)\sin x\,dx$. Let $u = \cos x$. It follows that $du = -\sin x\,dx$ and the integral is

$$-\int (1-u^2)du = -\left(u - \frac{u^3}{3}\right) + C = \boxed{-\cos x + \frac{\cos^3 x}{3} + C}.$$

<u>Check</u> The derivative of the proposed answer is $\sin x - \cos^2 x \sin x$ $= \sin x(1-\cos^2 x)\checkmark$.

4. **Find** $\displaystyle\int \sin^2 x\,dx$.

The integral is equal to

$$\frac{1}{2}\int (1-\cos 2x)\,dx = \boxed{\frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C}$$

5. **Find** $\displaystyle\int \sec^3 x\,dx$. **Check your answer.**

Let $u = \sec x$ and $dv = \sec^2 x\,dx$. It follows that $du = \sec x \tan x\,dx$ and $v = \tan x$. The integration by parts formula gives:

$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|.$$

Add $\int \sec^3 x\,dx$ to both sides to see that

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Divided by 2 to conclude

$$\boxed{\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C}$$

Check The derivative of the proposed answer is

$$\frac{1}{2}(\sec x \sec^2 x + \sec x \tan^2 x + \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}) = \frac{1}{2}\left(\sec^3 x + \sec x(1+\tan^2 x)\right)$$

$$= \sec^3 x \checkmark$$

6. **Find** $\int \frac{-2x^2-2x+1}{x^2(x^2+1)}\,dx$. **Check your answer.**

Use the technique of partial fractions to see that

$$\frac{-2x^2-2x+1}{x^2(x^2+1)} = \frac{-2}{x} + \frac{1}{x^2} + \frac{2x-3}{x^2+1}.$$

The integral is equal to

$$\boxed{-2\ln|x| - \frac{1}{x} + \ln(x^2+1) - 3\arctan x + C.}$$

7. **What is the limit of the sequence whose** n^{th} **term is** $a_n = \left(\frac{n+5}{n}\right)^n$ **?**

We see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+5}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = \boxed{e^5}.$$

I used $\lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$.

8. **Does the integral** $\int_2^\infty \frac{x}{x^6+1}\,dx$ **converge? Explain your answer very thoroughly.**

We see that $0 \le \frac{x}{x^6+1}$, for all x under consideration. There are only two possibilities. Either $\int_2^\infty \frac{x}{x^6+1}\,dx$ is a finite number or $\int_2^\infty \frac{x}{x^6+1}\,dx$ goes to infinity. We see that $\frac{x}{x^6+1} \le \frac{x}{x^6} = \frac{1}{x^5}$. It follows that

$$\int_2^\infty \frac{x}{x^6+1}\,dx = \lim_{b\to\infty} \int_2^b \frac{x}{x^6+1}\,dx \le \lim_{b\to\infty} \int_2^b \frac{1}{x^5}\,dx = \lim_{b\to\infty} \left.\frac{1}{-4x^4}\right|_2^b$$

$$= \lim_{b\to\infty} \frac{1}{-4b^4} - \frac{1}{-4(2)^4} = \frac{1}{64}.$$

Thus, $\int_2^\infty \frac{x}{x^6+1}\,dx$ converges to some finite non-negative number which is at most $\frac{1}{64}$.

9. **Consider the sequence**

$$\begin{aligned} a_1 &= \sqrt{6} \\ a_2 &= \sqrt{6+\sqrt{6}} \\ a_3 &= \sqrt{6+\sqrt{6+\sqrt{6}}} \\ a_4 &= \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}} \\ &\vdots \end{aligned}$$

Assume that you know that the sequence converges. Find the limit of the sequence. Explain your work very thoroughly.

Let L be the name of $\lim\limits_{n\to\infty} a_n$. We see that $a_{n+1} = \sqrt{6+a_n}$. Take the limit of each side to see that $L = \sqrt{L+6}$. Square both sides to get $L^2 = L+6$, or $L^2 - L - 6 = 0$. Factor to get $(L-3)(L+2) = 0$. It follows that L must equal -2 or 3. However, all of the members of our sequence are at least 0; so $L \ge 0$. Thus, $L \ne -2$ and $\boxed{L=3}$.

10. **Trapezoidal rule says that** $\int_a^b f(x)dx = T_n + E_n$ **. The formula for** T_n **is not needed in this problem. The value of** E_n **is** $E_n = \frac{-(b-a)^3}{12n^2} f''(c)$ **for some** c **between** a **and** b **. Suppose that Trapezoidal rule is used to approximate** $\int_1^3 \frac{1}{x}\,dx$ **, with** $n = 10$ **. Find a decent upper bound for** $|E_n|$ **. Explain your work very thoroughly.**

We have $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, and $f''(x) = 2x^{-3}$. So, if $n = 10$, then

$$|E_n| = |\frac{-(b-a)^3}{12n^2} f''(c)| = |\frac{-(3-1)^3}{12(10)^2} \frac{2}{c^3}| = \frac{8}{1200} \frac{2}{c^3}.$$

If $1 \le c \le 3$, then $\frac{1}{3^3} \le \frac{1}{c^3} \le \frac{1}{1}$, thus,

$$|E_n| \le \frac{16}{1200} = \frac{1}{75}.$$