Math 142, Exam 3, Spring 2011

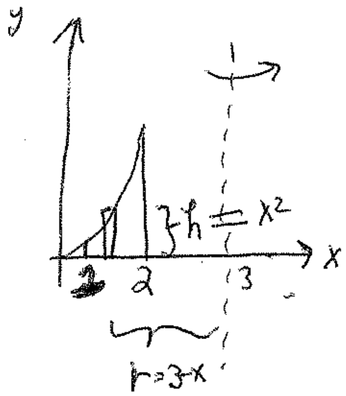
Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones. I will post the solutions on my website.

1. (7 points) **Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, the x-axis, $x = 1$, and $x = 2$ about the line $x = 3$.**

Chop the x-axis from $x = 1$ to $x = 2$, as shown. Over each small piece of the x-axis, draw a rectangle.



Spin the rectangle. Get a shell

of volume $2\pi rht$ where $t = dx$, $r = 3 - x$, and $h = x^2$

The volume of the shell with x-coordinate x is $2\pi(3-x)x^2dx$. The volume of the solid is

$$2\pi \int_1^2 (3x^2 - x^3)dx = 2\pi \left(x^3 - \frac{x^4}{4} \right)\Big|_1^2 = 2\pi \left((8-4) - \left(1 - \frac{1}{4}\right)\right)$$

$$= 2\pi \left(4 - \frac{3}{4}\right) = \boxed{\frac{13\pi}{2}}$$

2. (7 points) **Does the series** $\sum_{n=2}^{\infty} \frac{1+\sin n}{10^n}$ **converge? Justify your answer very thoroughly. Use complete sentences.**

We compare $\sum_{n=2}^{\infty} \frac{1+\sin n}{10^n}$ to $\sum_{n=2}^{\infty} \frac{2}{10^n}$. We know that $\sum_{n=2}^{\infty} \frac{2}{10^n}$ is a geometric series with ratio $\frac{1}{10}$. This ratio is between -1 and 1. Thus, $\sum_{n=2}^{\infty} \frac{2}{10^n}$ converges. We see that $0 \leq \frac{1+\sin n}{10^n} \leq \frac{2}{10^n}$. The comparison test yields that $\sum_{n=2}^{\infty} \frac{1+\sin n}{10^n}$ also converges.

3. (7 points) **Does the series $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ converge? Justify your answer very thoroughly. Use complete sentences.**

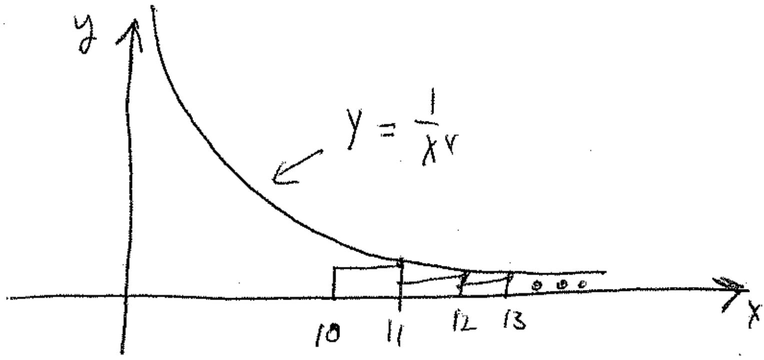
We compare $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ to $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$. The series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is the p-series with $p = \frac{1}{2} \leq 1$. Thus, we know that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges. We see that $0 < n-1 < n$. Divide by $(n-1)\sqrt{n}$ to see that $\frac{1}{\sqrt{n}} < \frac{\sqrt{n}}{n-1}$. Apply the comparison test to see that $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges.

4. (7 points) **Does the series $\sum_{n=2}^{\infty} (1-\frac{1}{n})^n$ converge? Justify your answer very thoroughly. Use complete sentences.**

We know that $\lim_{n\to\infty} (1-\frac{1}{n})^n = e^{-1}$. Thus, the individual terms of the seies $\sum_{n=2}^{\infty} (1-\frac{1}{n})^n$ go to something other than zero. Use the Individual Term Test for Divergence to see that the series $\sum_{n=2}^{\infty} (1-\frac{1}{n})^n$ diverges.

5. (7 points) **Estimate the distance between $\sum_{n=2}^{10} (\frac{1}{n^4})$ and $\sum_{n=2}^{\infty} (\frac{1}{n^4})$. I want your estimate to be close to, but larger than the exact distance. Justify your answer very thoroughly. Use complete sentences.**

The distance between $\sum_{n=2}^{10} (\frac{1}{n^4})$ and $\sum_{n=2}^{\infty} (\frac{1}{n^4})$ is $\sum_{n=11}^{\infty} (\frac{1}{n^4})$. Look at the picture:



The area inside the boxes is $\sum_{n=11}^{\infty}\left(\frac{1}{n^4}\right)$. The area inside the boxes is less than the area under the curve. Thus,

$$\sum_{n=11}^{\infty}\left(\frac{1}{n^4}\right) \leq \int_{10}^{\infty}\frac{1}{x^4}dx = \lim_{b\to\infty}\left.\frac{1}{-3x^3}\right|_{10}^{b} = \lim_{b\to\infty}\frac{1}{-3b^3}+\frac{1}{3000}=\frac{1}{3000}.$$

Thus,

$$\boxed{\sum_{n=2}^{10}\left(\frac{1}{n^4}\right) \text{ approximates } \sum_{n=2}^{\infty}\left(\frac{1}{n^4}\right) \text{ with an error at most } \frac{1}{3000}.}$$

6. (7 points) **Express** 3.4174174174... **as a ratio of integers. Justify your answer very thoroughly. Use complete sentences.**

Let $s = 3.4174174174\ldots$. Observe that

$$\begin{aligned} 1000s - s &= 3417.417417417\ldots \\ &- \quad 3.417417417\ldots \end{aligned}$$

So, $999s = 3414$ and $\boxed{s = \frac{3414}{999}}$

7. (8 points) **Justify your answer very thoroughly. Use complete sentences.** Consider the sequence $\{a_n\}$ with $a_1 = 0$ and $a_n = \sqrt{20 + a_{n-1}}$ for $n \geq 2$.
   (a) Show that $0 \leq a_n \leq 10$ for all n.
   (b) Show that $\{a_n\}$ is an increasing sequence.
   (c) Explain why the sequence $\{a_n\}$ converges.
   (d) Find the limit of the sequence $\{a_n\}$.

(a) We see that $0 \leq a_1 \leq 10$. Assume, **by induction**, that $0 \leq a_n \leq 10$ for some fixed n. Add 20 and take the square root to see that $20 \leq 20 + a_n \leq 30$ and $\sqrt{20} \leq \sqrt{20 + a_n} \leq \sqrt{30}$. We see that $0 \leq \sqrt{20}$, $a_{n+1} = \sqrt{20 + a_n}$, and $\sqrt{30} \leq 10$. Thus, we have shown that if $0 \leq a_n \leq 10$, for some fixed n, then the inequalities $0 \leq a_{n+1} \leq 10$ also hold. We apply Mathematical Induction to conclude that $0 \leq a_n \leq 10$ for all n.

(b) We see that $a_1 < a_2$. Assume, **by induction**, that $a_{n-1} \leq a_n$ for some fixed n. Add 20 and take square root to see that $20 + a_{n-1} \leq 20 + a_n$, and $\sqrt{20 + a_{n-1}} \leq \sqrt{20 + a_n}$. On the other hand, $\sqrt{20 + a_{n-1}} = a_n$ and $\sqrt{20 + a_n} = a_{n+1}$. We have shown that the sequence $\{a_n\}$ starts out as an increasing sequence; and if it is increasing until a certain place, then it is increasing one spot betond the certain place. We apply the principal of Mathematical Induction to conclude that the sequence is always increasing.

(c) We have shown that $\{a_n\}$ is an increasing bounded sequence. The Completeness Axiom guarantees that every increasing bound sequence of real numbers converges. We conclude that $\{a_n\}$ converges. Let $L = \lim\limits_{n\to\infty} a_n$.

(d) Apply $\lim\limits_{n\to\infty}$ to both sides of $a_n = \sqrt{20 + a_{n-1}}$ to see that $\lim\limits_{n\to\infty} a_n = \sqrt{20 + \lim\limits_{n\to\infty} a_{n-1}}$. In other words, $L = \sqrt{20 + L}$. Square both sides to see that $L^2 = 20 + L$, or $L^2 - L - 20 = 0$, or $(L-5)(L+4) = 0$. Thus, $L = 5$ or $L = -4$. Every term in the sequence is at least 0. The limit of the sequence can not possibly negative. Thus, $L = 5$.