Math 142, Exam 3, Spring 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Find** $\int \frac{x^2-6x+12}{(x-2)^3}dx$ **. Please make sure that your answer is correct.**

We apply the technique of partial fractions and look for numbers A, B, and C with

$$\frac{x^2-6x+12}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}.$$

Multiply both sides by $(x-2)^3$ to see that

$$x^2-6x+12 = A(x-2)^2 + B(x-2) + C.$$

Equate the corresponding coefficients:

$$x^2-6x+12 = A(x^2-4x+4) + B(x-2) + C = Ax^2 + (-4A+B)x + 4A - 2B + C.$$

So

$$1 = A, \quad -6 = -4A + B, \quad 12 = 4A - 2B + C.$$

Thus, $A = 1$, $B = -2$, and $C = 4$. We check this much:

$$\frac{1}{x-2} + \frac{-2}{(x-2)^2} + \frac{4}{(x-2)^3} = \frac{(x^2-4x+4) - 2(x-2) + 4}{(x-2)^3} = \frac{x^2-6x+12}{(x-2)^3},$$

as expected. Now we do the integral

$$\int \frac{x^2-6x+12}{(x-2)^3}dx = \int \left(\frac{1}{x-2} + \frac{-2}{(x-2)^2} + \frac{4}{(x-2)^3}\right) dx$$

$$= \boxed{\ln|x-2| + \frac{2}{x-2} - \frac{2}{(x-2)^2} + C}.$$

2. **Find** $\displaystyle\int_{-4}^{7} \frac{1}{(x-5)^2}dx$ **.**

There is a pretty picture of the area that this integral represents elsewhere in the solution set. The function $\frac{1}{(x-5)^2}$ misbehaves at $x = 5$. We have to treat $x = 5$ carefully.

$$\int_{-4}^{7} \frac{1}{(x-5)^2}dx = \lim_{b\to 5^-}\int_{-4}^{b} \frac{1}{(x-5)^2}dx + \lim_{a\to 5^+}\int_{a}^{7} \frac{1}{(x-5)^2}dx$$

$$= \lim_{b\to 5^-} \left.\frac{-1}{(x-5)}\right|_{-4}^{b} + \lim_{a\to 5^+} \left.\frac{-1}{(x-5)}\right|_{a}^{7}$$

$$= \lim_{b\to 5^-} \frac{-1}{(b-5)} - \frac{-1}{(-4-5)} + \frac{-1}{(7-5)} - \lim_{a\to 5^+} \frac{-1}{(a-5)} = +\infty - \frac{1}{9} - \frac{1}{2} + \infty.$$

The integral diverges to $+\infty$.

3. **Approximate** $\sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ **with an error at most** $\frac{1}{100}$**. Please explain what you are doing and why.**

The error that is introduced when $\sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ is approximated by $\sum\limits_{k=1}^{N} \frac{1}{k^3}$ is

$$\left|\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{N} \frac{1}{k^3}\right| = \sum_{k=N+1}^{\infty} \frac{1}{k^3}.$$

Look at the picture that appears elsewhere in this solution set to see that

$$\sum_{k=N+1}^{\infty} \frac{1}{k^3} \le \int_{N}^{\infty} \frac{1}{x^3} = \lim_{b\to\infty} \left.-\frac{1}{2x^2}\right|_{N}^{b} = \lim_{b\to\infty} -\frac{1}{2b^2} + \frac{1}{2N^2} = \frac{1}{2N^2}.$$

We make $\frac{1}{2N^2} \le \frac{1}{100}$. We make $50 \le N^2$. This occurs when $8 \le N$. We conclude that

$$\sum_{k=1}^{8} \frac{1}{k^3} \text{ approximates } \sum_{k=1}^{\infty} \frac{1}{k^3} \text{ with an error at most } 1/100.$$

4. **Find the Taylor polynomial** $P_3(x)$ **for** $f(x) = \ln(x)$ **centered about** $a = 1$ **.**
We see that

$$\begin{array}{cc} f(x) = \ln(x) & f(1) = 0 \\ f'(x) = \frac{1}{x} & f'(1) = 1 \\ f''(x) = -\frac{1}{x^2} & f''(1) = -1 \\ f''(x) = 2\frac{1}{x^3} & f'''(1) = 2 \end{array}$$

The Taylor polynomial $P_3(x)$ is equal to

$$P_3(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{3!};$$

hence in our problem

$$\boxed{P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{3!}.}$$

Of course, this is the same as $P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ and this is not shocking because we saw in class that

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots \quad \text{for } 0 < x < 2\,.$$

5. **Where does the power series** $f(x) = \sum_{n=1}^{\infty} \frac{(x-5)^n}{n2^n}$ **converge? Explain carefully. Be sure to account for all real numbers** x **.**
We use the ratio test. Let

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{|x-5|^{n+1}}{(n+1)2^{n+1}}}{\frac{|x-5|^n}{n2^n}} = \lim_{n\to\infty} \frac{|x-5|^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{|x-5|^n} = \lim_{n\to\infty} \frac{n|x-5|}{(n+1)2}$$

$$= \lim_{n\to\infty} \frac{|x-5|}{(1+\frac{1}{n})2} = \frac{|x-5|}{2}$$

If $\frac{|x-5|}{2} < 1$, then $f(x)$ converges. If $1 < \frac{|x-5|}{2}$, then $f(x)$ diverges. We will need to use another test when $\frac{|x-5|}{2} = 1$.

We see that $\frac{|x-5|}{2} < 1$ is the same as $|x-5| < 2$, which is the same as $-2 < x-5 < 2$, which is the same as $3 < x < 7$.

We see that $1 < \frac{|x-5|}{2}$ is the same as $2 < |x-5|$, which is the same as $x-5 < -2$ or $2 < x-5$, which is the same as $x < 3$ or $7 < x$.

When $x = 7$, $f(7) = \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, and this is the harmonic series which diverges.

When $x = 3$, $f(3) = \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, and this is (minus) the alternating harmonic series which converges.

We conclude that $f(x)$ converges for $3 \leq x < 7$ and $f(x)$ diverges everywhere else.