Math 142, Exam 3, Solutions, Fall 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. This work must be coherent and correct. $\boxed{CIRCLE}$ your answer. **No Calculators or Cell phones.**

The solutions will be posted later today.

1. (8 points) **Find the volume of the solid that is obtained by revolving the region bounded by $y^2 = x$ and $x - y = 2$ about the line $y = -6$. You must draw a meaningful picture. (There is no need for you to do the final arithmetic. That is, you may stop as soon as you have plugged the endpoints into an anti-derivative.)**

The picture appears elsewhere. The intersection points are found by solving $y^2 - y - 2 = 0$ and this is $(y-2)(y+1) = 0$. So $y = 2$ and $y = -1$. The intersection points are $(1, -1)$ and $(4, 2)$. Notice that these points satisfy both equations $y^2 = x$ and $x - y = 2$. Chop the y-axis from $y = -1$ to $y = 2$. Consider the rectangle with y-coordinate y. Revolve this rectangle about the line $y = -6$ to obtain a shell of volume $2\pi rht$, where $t = dy$, $r = y + 6$, and $h = y + 2 - y^2$. The volume of the shell is $2\pi rht = 2\pi(y+6)(y+2-y^2)dy$. The volume of the solid is

$$2\pi \int_{-1}^{2} (y+6)(y+2-y^2)dy = 2\pi \int_{-1}^{2} (-y^3 - 5y^2 + 8y + 12)dy$$

$$= 2\pi\left(-\tfrac{y^4}{4} - \tfrac{5y^3}{3} + 4y^2 + 12y\right)\Big|_{-1}^{2}$$

$$= \boxed{2\pi\left(-\tfrac{2^4}{4} - \tfrac{5\cdot 2^3}{3} + 4\cdot 2^2 + 12(2) - \left(-\tfrac{(-1)^4}{4} - \tfrac{5(-1)^3}{3} + 4(-1)^2 + 12(-1)\right)\right)}$$

2. (7 points) **Consider a solid S whose base in the xy plane is the region bounded by $y^2 = x$ and $x - y = 2$. Each cross-section of S perpendicular to the y-axis is a square. Find the volume of S. You must draw a meaningful picture. (There is no need for you to do the final arithmetic. That is, you may stop as soon as you have plugged the endpoints into an anti-derivative.)**

The picture appears elsewhere. The intersection points are still $(1, -1)$ and $(4, 2)$. Chop the y-axis from $y = -1$ to $y = 2$. Consider the slice of S with

y-coordinate y. This slice is a square with thickness. The volume of the slice is the area of the square times the thickness and this is s^2t, where $t = dy$ and $s = y + 2 - y^2$. So the volume of the slice is

$$s^2t = (y + 2 - y^2)^2 dy.$$

The volume of the solid is

$$\int_{-1}^{2}(y + 2 - y^2)^2 dy = \int_{-1}^{2}(y^2 + 4y - 2y^3 + 4 - 4y^2 + y^4)dx$$

$$= \int_{-1}^{2}(4y - 2y^3 + 4 - 3y^2 + y^4)dx = \left(2y^2 - \tfrac{1}{2}y^4 + 4y - y^3 + \tfrac{1}{5}y^5\right)\Big|_{-1}^{2}$$

$$= \boxed{\left((8 - \tfrac{1}{2}2^4 + 8 - 8 + \tfrac{1}{5}2^5) - (2 - \tfrac{1}{2} - 4 + 1 - \tfrac{1}{5})\right)}$$

3. (7 points) **Consider the sequence defined by $a_1 = 2$ and $a_{n+1} = \frac{1}{4-a_n}$. Justify your answers very thoroughly. Write in complete sentences.**
   (a) **Prove that $0 < a_n \le 2$ for all positive integers n.**
   (b) **Prove that $a_{n+1} \le a_n$ for all positive integers n.**
   (c) **State the Completeness Axiom and draw a conclusion about the sequence $\{a_n\}$ from the Completeness Axiom.**
   (d) **Find the limit of the sequence $\{a_n\}$.**

(a) We use the technique of Mathematical Induction. We see that $a_1 = 2$ and therefore, $0 < a_1 \le 2$. Assume **BY INDUCTION** that $0 < a_{n-1} \le 2$ for some **FIXED** n. Multiply by -1 to see $-2 \le -a_{n-1} < 0$. Add 4 to see $2 \le 4 - a_{n-1} < 4$; that is $2 \le 4 - a_{n-1}$ and $4 - a_{n-1} < 4$. Divide the first inequality by the positive number $2(4 - a_{n-1})$ to obtain $\frac{1}{4-a_{n-1}} \le \frac{1}{2}$. Divide the second inequality by the positive number $(4 - a_{n-1})4$ to see $\frac{1}{4} < \frac{1}{4-a_{n-1}}$. Put the inequalities back together to see: $\frac{1}{4} < \frac{1}{4-a_{n-1}} \le \frac{1}{2}$. We have shown that

$$0 < a_{n-1} \le 2 \implies \frac{1}{4} < \frac{1}{4 - a_{n-1}} \le \frac{1}{2}.$$

Obviously, $\frac{1}{4-a_{n-1}} = a_n$, $0 < \frac{1}{4}$ and $\frac{1}{2} \le 2$; so,

$$0 < a_{n-1} \le 2 \implies 0 < a_n \le 2.$$

We saw that $0 < a_n \le 2$ for $n = 1$. We proved that if $0 < a_{n-1} \le 2$ for some FIXED n, then $0 < a_n \le 2$ also holds for that one FIXED n. We apply the

Principle of Mathematical Induction to conclude that $0 < a_n \le 2$ for ALL positive integers n.

(b) We use the technique of Mathematical Induction. We see that $a_1 = 2$ and $a_2 = \frac{1}{2}$; so $a_2 \le a_1$. Assume **BY INDUCTION** that $a_n \le a_{n-1}$ for some **FIXED** n. Add $-a_n - a_{n-1}$ to both sides to see $-a_{n-1} \le -a_n$. Add 4 to both sides to see: $4 - a_{n-1} \le 4 - a_n$. Both numbers are positive because part (1) shows that $a_n \le 2$ for all n. Divide both sides by the positive number $(4-a_{n-1})(4-a_n)$ to obtain $\frac{1}{4-a_n} \le \frac{1}{4-a_{n-1}}$ and this is $a_{n+1} \le a_n$. Thus

$$a_n \le a_{n-1} \implies a_{n+1} \le a_n.$$

We saw that $a_{n+1} \le a_n$ for $n = 1$. We proved that if $a_n \le a_{n-1}$ for some FIXED n, then $a_{n+1} \le a_n$ also holds for that one FIXED n. We apply the Principle of Mathematical Induction to conclude that $a_{n+1} \le a_n$ for ALL positive integers n.

(c) The completeness axiom says that every decreasing bounded sequence of real numbers has a limit. We showed in (1) and (2) that $\{a_n\}$ is an decreasing bounded sequence of real numbers. We conclude that $\lim\limits_{n\to\infty} a_n$ exists. Let $L = \lim\limits_{n\to\infty} a_n$.

(d) Take $\lim\limits_{n\to\infty}$ of both sides of $a_{n+1} = \frac{1}{4-a_n}$ to conclude that

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{4 - \lim\limits_{n\to\infty} a_n};$$

that is, $L = \frac{1}{4-L}$; so $L(4-L) = 1$ or $-L^2 + 4L = 1$. We use the quadratic formula to solve $0 = L^2 - 4L + 1$. We obtain $L = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$. We know that L can not be more than 2 because every term in the sequence is less than or equal to 2. So $L \ne 2 + \sqrt{3}$ and hence L does equal $2 - \sqrt{3}$.

4. (7 points) **Estimate the distance between $\sum\limits_{k=1}^{100} \frac{1}{k^4}$ and $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$. Your answer should be in the form "The distance between $\sum\limits_{k=1}^{100} \frac{1}{k^4}$ and $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ is less than xxx", where xxx is some small positive number that you have calculated. Justify your answer very thoroughly. Write in complete sentences. You must draw a meaningful picture.**

The picture appears in a separate file. The distance between $\sum\limits_{k=1}^{100} \frac{1}{k^4}$ and $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ is equal to

$$\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{100} \frac{1}{k^4} = \sum_{k=101}^{\infty} \frac{1}{k^4} = \text{the area inside the boxes} \le \text{the area under the curve}$$

$$= \int_{100}^{\infty} \frac{1}{x^4} dx = \lim_{b\to\infty} \frac{1}{-3x^3}\Big|_{100}^{b} = \lim_{b\to\infty} \frac{1}{-3b^3} + \frac{1}{3(100)^3} = \frac{1}{3\times 10^6}.$$

We conclude that

$$\boxed{\text{The distance between } \sum_{k=1}^{100} \frac{1}{k^4} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^4} \text{ is less than } \frac{1}{3\times 10^6}.}$$

5. (7 points) **Consider the series** $\frac{2}{5} - (\frac{2}{5})^2 + (\frac{2}{5})^3 - (\frac{2}{5})^4 + \dots$ **. Justify your answer very thoroughly. Write in complete sentences.**

(a) **Find a closed formula for the** n^{th} **partial sum**

$$s_n = \tfrac{2}{5} - (\tfrac{2}{5})^2 + (\tfrac{2}{5})^3 - (\tfrac{2}{5})^4 + \cdots + (-1)^{n+1}(\tfrac{2}{5})^n$$

of this series.

We see that $s_n - (-\frac{2}{5})s_n$ is equal to

$$\begin{aligned} &\tfrac{2}{5} - (\tfrac{2}{5})^2 + (\tfrac{2}{5})^3 - (\tfrac{2}{5})^4 + \cdots + (-1)^{n+1}(\tfrac{2}{5})^n \\ &\quad + (\tfrac{2}{5})^2 - (\tfrac{2}{5})^3 + (\tfrac{2}{5})^4 + \cdots + (-1)^n(\tfrac{2}{5})^n + (-1)^{n+1}(\tfrac{2}{5})^{n+1}. \end{aligned}$$

Thus, $(1+\frac{2}{5})s_n = \frac{2}{5} + (-1)^{n+1}(\frac{2}{5})^{n+1}$ and

$$\boxed{s_n = \frac{\frac{2}{5} + (-1)^{n+1}(\frac{2}{5})^{n+1}}{\frac{7}{5}}}$$

(b) **Find the sum of the entire series.**

The sum of the series is the limit of the sequence of partial sums and this equals $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{\frac{2}{5}+(-1)^{n+1}(\frac{2}{5})^{n+1}}{\frac{7}{5}} = \frac{\frac{2}{5}}{\frac{7}{5}} = \frac{2}{7}$. Thus,

$$\boxed{\text{the sum of the series } \tfrac{2}{5} - (\tfrac{2}{5})^2 + (\tfrac{2}{5})^3 - (\tfrac{2}{5})^4 + \dots \text{ is } \tfrac{2}{7}.}$$

Of course, when $a = \frac{2}{5}$ and $r = -\frac{2}{5}$, then $\frac{a}{1-r}$ is also equal to $\frac{\frac{2}{5}}{1+\frac{2}{5}} = \frac{2}{7}$.

6. (7 points) **Consider the series** $\ln\frac{2}{3}+\ln\frac{3}{4}+\ln\frac{4}{5}+\ln\frac{5}{6}\dots$ **. Justify your answer very thoroughly. Write in complete sentences.**

(a) **Find a closed formula for the** n^{th} **partial sum**

$$s_n=\ln\tfrac{2}{3}+\ln\tfrac{3}{4}+\ln\tfrac{4}{5}+\ln\tfrac{5}{6}\cdots+\ln\tfrac{n+1}{n+2}$$

of this series.

We see that

$$s_n=\begin{cases}(\ln 2-\ln 3^*)+(\ln 3^*-\ln 4^{**})+(\ln 4^{**}-\ln 5^{***})+(\ln 5^{***}-\ln 6^{****})+^{****}\dots^{\dagger\dagger}\\ +(\ln(n)^{\dagger\dagger}-\ln(n+1)^{\dagger})+(\ln(n+1)^{\dagger}-\ln(n+2)).\end{cases}$$

Thus, $\boxed{s_n=\ln 2-\ln(n+2)}$.

(b) **Find the sum of the entire series.**

The sum of the series is the limit of the sequence of partial sums and this equals $\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}(\ln 2-\ln(n+2))=-\infty$. We conclude that

$$\boxed{\text{the series }\ln\frac{2}{3}+\ln\frac{3}{4}+\ln\frac{4}{5}+\ln\frac{5}{6}\dots\text{ diverges to }-\infty}.$$

7. (7 points) **Does the series** $\sum\limits_{k=1}^{\infty}\frac{2k}{k^2+1}$ **converge? Justify your answer very thoroughly. Write in complete sentences.**

We use the Limit Comparison Test. The series $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ is the Harmonic series; this series is known to diverge. Think of the original series as $\sum a_k$ and the Harmonic series as $\sum b_k$. We compute

$$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{\frac{2k}{k^2+1}}{\frac{1}{k}}=\lim_{k\to\infty}\frac{2k^2}{k^2+1}=\lim_{k\to\infty}\frac{2}{1+\frac{1}{k^2}}=2.$$

We see that 2 is a number; 2 is not zero or ∞. The Limit Comparison Test ensures that the series $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ and $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ both converge or both diverge. We have already seen that $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ diverges. We conclude that

$$\boxed{\sum_{k=1}^{\infty}\frac{2k}{k^2+1}\text{ also diverges.}}$$