Math 142, Exam 1, SOLUTIONS, Fall 2015

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Find** $\int \sin^4 x \cos^5 x dx$ **. Please check your answer.**

Save one $\cos x$. Use $\sin^2 x + \cos^2 x = 1$ to convert the rest of the $\cos x$ into $\sin x$. Let $u = \sin x$. It follows that $du = \cos x dx$. We see that

$$\int \sin^4 x \cos^5 x dx = \int \sin^4 x (\cos^2 x)^2 \cos x dx = \int u^4 (1-u^2)^2 du$$

$$= \int (u^4 - 2u^6 + u^8) du = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C = \boxed{\frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C}.$$

Check. The derivative of the proposed answer is

$$\sin^4 x \cos x - 2\sin^6 x \cos x + \sin^8 x \cos x = \sin^4 x \cos x (1 - 2\sin^2 x + \sin^4 x)$$

$$= \sin^4 x \cos x (1 - \sin^2 x)^2 = \sin^4 x \cos x (\cos^2 x)^2 = \sin^4 x \cos^5 x. \checkmark$$

2. **Find** $\int e^{4x} \sin x dx$ **. Please check your answer.**

Use integration by parts. Let $u = e^{4x}$ and $dv = \sin x\, dx$. Calculate that $du = 4e^{4x} dx$ and $v = -\cos x$. It follows that

$$\int e^{4x} \sin x dx = -e^{4x} \cos x + 4 \int e^{4x} \cos x.$$

Use integration by parts again. Let $u = e^{4x}$ and $dv = \cos x\, dx$. Calculate that $du = 4e^{4x} dx$ and $v = \sin x$. It follows that

$$\int e^{4x} \sin x dx = -e^{4x} \cos x + 4 \left(e^{4x} \sin x - 4 \int e^{4x} \sin x\, dx \right).$$

Add $16\int e^{4x}\sin x\,dx$ to both sides of the equation to obtain

$$17\int e^{4x}\sin x dx = -e^{4x}\cos x + 4e^{4x}\sin x + C.$$

Divide both sides by 17 to conclude

$$\int e^{4x}\sin x dx = \frac{1}{17}\left(-e^{4x}\cos x + 4e^{4x}\sin x\right) + K = \boxed{\frac{e^{4x}}{17}(-\cos x + 4\sin x) + K}.$$

Check. The derivative of the proposed answer is

$$\frac{e^{4x}}{17}(\sin x + 4\cos x) + \frac{4e^{4x}}{17}(-\cos x + 4\sin x)$$
$$= \frac{e^{4x}}{17}(\sin x + 4\cos x - 4\cos x + 16\sin x) = e^{4x}\sin x \checkmark$$

3. **Find** $\int \frac{3x^2 - 3x + 2}{x^3 + x}dx$. **Please check your answer.**

We use the technique of partial fractions. The demoninator factors as $x(x^2+1)$. We look for A, B, and C with

$$\frac{3x^2 - 3x + 2}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Multiply both sides by $x(x^2+1)$ to obtain

$$3x^2 - 3x + 2 = A(x^2+1) + (Bx + C)x = (A + B)x^2 + Cx + A.$$

Equate the corresponding coefficients to see that

$$3 = A + B, \quad -3 = C, \quad 2 = A.$$

Calculate that $B = 1$. We seem to have shown that

$$\frac{3x^2 - 3x + 2}{x^3 + x} = \frac{2}{x} + \frac{x - 3}{x^2 + 1}.$$

We verify this claim before going any further. The right side is

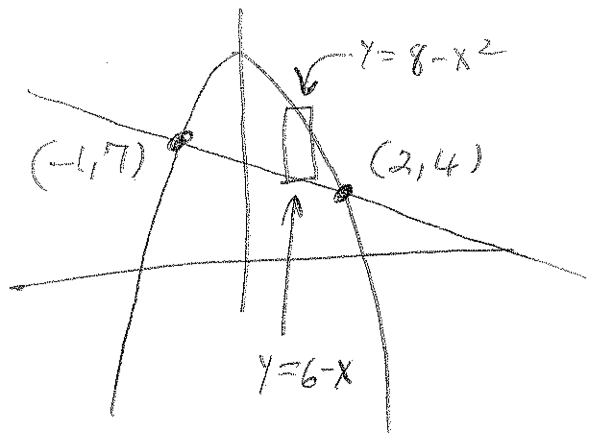
$$\frac{2x^2 + 2 + x^2 - 3x}{x(x^2+1)} = \frac{3x^2 - 3x + 2}{x^3 + x},$$

as claimed. Now we compute

$$\int \frac{3x^2 - 3x + 2}{x^3 + x}dx = \int\left(\frac{2}{x} + \frac{x - 3}{x^2 + 1}\right)dx$$
$$= \boxed{2\ln x + \frac{1}{2}\ln(x^2+1) - 3\arctan x + C}.$$

4. **Find the area between $y + x^2 = 8$ and $x + y = 6$. Please draw a meaningful picture.**

We know that $y = 8 - x^2$ is an upside down parabola which has been lifted up by 8 and $x+y=6$ is the line that passes through $(6,0)$ and $(0,6)$. The intersection occurs when $x+(8-x^2)=6$; so, $0 = x^2 - x - 2 = (x-2)(x+1)$. The intersection occurs when $x = 2$ and when $x = -1$. The intersection points are $(2,4)$ and $(-1,7)$. The picture is



The rectangle that we have drawn has area

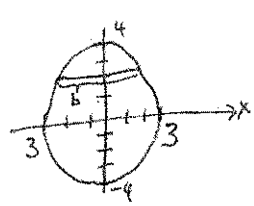
$$[(8-x^2)-(6-x)]dx = (2+x-x^2)dx.$$

The region has area

$$\int_{-1}^{2} (2+x-x^2)dx = \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-1}^{2} = \boxed{(4+2-\frac{8}{3}) - (-2+\frac{1}{2}+\frac{1}{3})}.$$

5. **Consider the solid whose base is bounded by $\frac{x^2}{9} + \frac{y^2}{16} = 1$ in the xy-plane. Each cross section of the solid perpendicular to the y-axis and perpendicular to the base is a square. Find the volume of the solid. Please draw a meaningful picture.**

The base of the solid is an ellipse:



Chop the y-axis from -4 to 4. Consider the slice of the solid with y-coordinate y. This slice has volume b^2t. We see that $\frac{b}{2}$ is the x-coordinate on the ellipse that corresponds to y; so, $\frac{b}{2} = 3\sqrt{1 - \frac{y^2}{16}}$ and $b = 6\sqrt{1 - \frac{y^2}{16}}$. The volume of the slice is

$$b^2t = 36(1 - \tfrac{y^2}{16})dy.$$

The volume of the solid is

$$36\int_{-4}^{4}\left(1 - \frac{y^2}{16}\right)dy = 36\left(y - \frac{y^3}{48}\right)\Bigg|_{-4}^{4} = \boxed{72\left(4 - \frac{4}{3}\right)}.$$