

Math 142, Exam 2, Solutions, Fall 2011

Write everything on the blank paper provided. You should **KEEP** this piece of paper. If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it - I will still grade your exam.

The exam is worth 50 points. SHOW your work. CIRCLE your answer. CHECK your answer whenever possible.

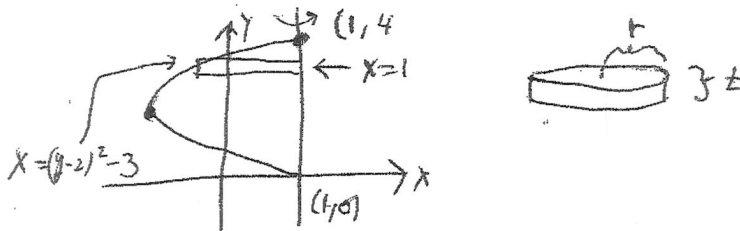
No Calculators or Cell phones.

1. (6 points) Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.

Let $f(x)$ be a function defined on the closed interval $a \leq x \leq b$. For each partition P of the closed interval $[a, b]$ (so, P is $a = x_0 \leq x_1 \leq \dots \leq x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum_{i=1}^n f(x_i^*)\Delta_i$.

2. (6 points) Consider the region bounded by $x + 3 = (y - 2)^2$ and $x = 1$. Revolve this region about the line $x = 1$. Find the volume of the resulting solid. You must draw a meaningful picture.

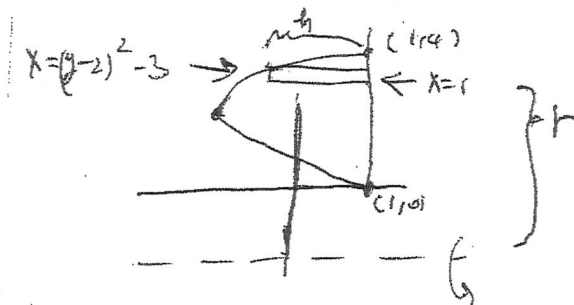
The intersection points occur when $4 = (y - 2)^2$; so $2 = y - 2$ or $-2 = y - 2$.



Chop the y -axis from $y = 0$ to $y = 4$. Consider the rectangle with y -coordinate y . Revolve the rectangle about $x = 1$. Get a disk of volume $\pi r^2 t$, with $t = dy$, and $r = 1 - ((y - 2)^2 - 3) = 1 - (y^2 - 4y + 4 - 3) = 4y - y^2$. The vol of the disk is $\pi r^2 t = \pi(4y - y^2)^2 dy$. The volume of the solid is

$$\pi \int_0^4 (16y^2 - 8y^3 + y^4) dy = \pi(16y^3/3 - 2y^4 + y^5/5)|_0^4 = \boxed{\pi(16(64)/3 - 2(4)^4 + 4^5/5)}$$

3. (6 points) Consider the region bounded by $x + 3 = (y - 2)^2$ and $x = 1$. Revolve this region about the line $y = -1$. Find the volume of the resulting solid. You must draw a meaningful picture.

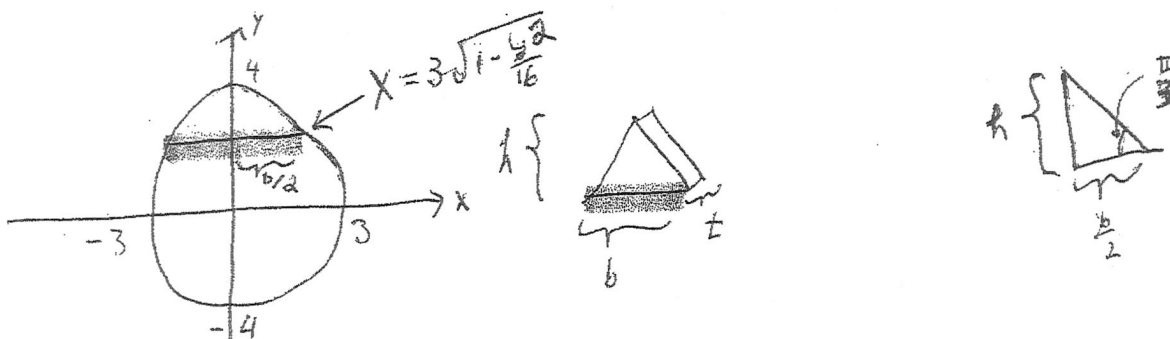


Chop the y -axis from $y = 0$ to $y = 4$. Consider the rectangle with y -coordinate y . Revolve the rectangle about $y = -1$. Get a shell of volume $2\pi rht$, with $t = dy$, $r = y + 1$ and $h = 1 - ((y - 2)^2 - 3) = 4y - y^2$. The volume of the shell is $2\pi rht = 2\pi(y + 1)(4y - y^2)dy$. The volume of the solid is

$$2\pi \int_0^4 (-y^3 + 3y^2 + 4y)dy = 2\pi(-y^4/4 + y^3 + 2y^2)|_0^4 = 2\pi(-64 + 64 + 32) = \boxed{64\pi}.$$

4. (6 points) Consider the solid whose base is bounded by $\frac{x^2}{9} + \frac{y^2}{16} = 1$ in the xy -plane. Each cross section of the solid perpendicular to the y -axis and perpendicular to the base is an equilateral triangle. Find the volume of the solid. You must draw a meaningful picture.

The base of the solid is an ellipse:



Chop the y -axis from -4 to 4 . Consider the slice of the solid with y -coordinate y . This slice has volume $(1/2)bht$. We have $t = dy$, $\frac{h}{(b/2)} = \frac{\text{Op}}{\text{Adj}} = \tan \frac{\pi}{3} = \sqrt{3}$. So, $h = \sqrt{3}\frac{b}{2}$. We see that $\frac{b}{2}$ is the x -coordinate on the ellipse that corresponds to y ; so, $\frac{b}{2} = 3\sqrt{1 - \frac{y^2}{16}}$. The volume of the slice is

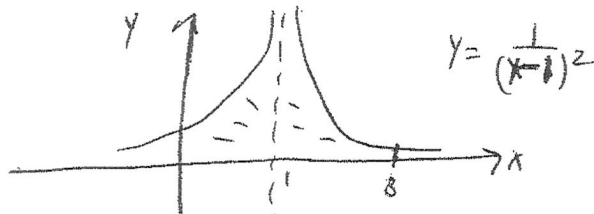
$$\frac{1}{2}bht = \sqrt{3}\frac{b}{2}t = \sqrt{3}\left(\frac{b}{2}\right)^2 t = 9\sqrt{3}\left(1 - \frac{y^2}{16}\right)dy.$$

The volume of the solid is

$$9\sqrt{3} \int_{-4}^4 \left(1 - \frac{y^2}{16}\right) dy = 9\sqrt{3} \left(y - \frac{y^3}{48}\right) \Big|_{-4}^4 = \boxed{18\sqrt{3} \left(4 - \frac{4}{3}\right)}.$$

5. (6 points) Find $\int_0^3 \frac{1}{(x-1)^2} dx$.

The function $f(x) = \frac{1}{(x-1)^2}$ goes to infinity as x gets near 1:



The answer will either be a positive number or $+\infty$. The integral is equal to

$$\begin{aligned} & \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{a \rightarrow 1^+} \int_a^3 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \frac{1}{-(x-1)} \Big|_0^b + \lim_{a \rightarrow 1^+} \frac{1}{-(x-1)} \Big|_a^3 \\ &= \lim_{b \rightarrow 1^-} \frac{1}{-(b-1)} - 1 + \lim_{a \rightarrow 1^+} \left(\frac{1}{-2} + \frac{1}{a-1} \right) \end{aligned}$$

Both limits $\lim_{b \rightarrow 1^-} \frac{1}{-(b-1)}$ and $\lim_{a \rightarrow 1^+} \frac{1}{a-1}$ equal $+\infty$. The answer is $\boxed{+\infty}$.

6. (6 points) Find $\int \cos^5 x dx$. You must check your answer.

Save one $\cos x$. Let $u = \sin x$. It follows that $du = \cos x dx$. The integral is equal to

$$\begin{aligned} \int (1 - \sin^2 x)^2 \cos x dx &= \int (1 - u^2)^2 du = \int (1 - 2u^2 + u^4) du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C \\ &= \boxed{\sin x - \frac{2}{3} \sin^3 x + \frac{\sin^5 x}{5} + C}. \end{aligned}$$

Check. The derivative of the proposed answer is

$$\cos x - 2 \sin^2 x \cos x + \sin^4 x \cos x = \cos x (1 - 2 \sin^2 x + \sin^4 x) = \cos x (1 - \sin^2 x)^2.$$

7. (7 points) **Find** $\int \sec x \tan^2 x dx$. **You must check your answer.**

Find $\int \sec x \tan^2 x dx$. **Check your answer.**

We do this integral by parts. Let $u = \tan x$ and $dv = \sec x \tan x dx$. We compute $du = \sec^2 x dx$ and $v = \sec x$. We have

$$\begin{aligned} \int \sec x \tan^2 x dx &= \sec x \tan x - \int \sec^3 x dx = \sec x \tan x - \int \sec x (\tan^2 x + 1) dx \\ &= \sec x \tan x - \int \sec x \tan^2 x dx - \int \sec x dx. \end{aligned}$$

Add $\int \sec x \tan^2 x dx$ to both sides

$$2 \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x dx = \sec x \tan x - \ln |\sec x + \tan x| + C.$$

Divide by 2 to see that

$$\boxed{\int \sec x \tan^2 x dx = \frac{1}{2}(\sec x \tan x - \ln |\sec x + \tan x|) + C.}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} &\frac{1}{2} \left(\sec x \sec^2 x + \sec x \tan x \tan x - \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \right) \\ &= \frac{1}{2} (\sec x \sec^2 x + \sec x \tan x \tan x - \sec x) \\ &= \frac{1}{2} \sec x (\sec^2 x + \tan^2 x - 1) = \frac{1}{2} \sec x (\tan^2 x + \tan^2 x). \checkmark \end{aligned}$$

8. (7 points) **Find** $\int \frac{4x^3 + 6x^2 + 4x + 8}{x^4 + 4x^2} dx$. **You must check your answer.**

We see that $x^4 + 4x^2 = x^2(x^2 + 4)$. Write

$$\frac{4x^3 + 6x^2 + 4x + 8}{x^4 + 4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 4}.$$

Multiply both sides by $x^4 + 4x^2$ to get

$$4x^3 + 6x^2 + 4x + 8 = Ax(x^2 + 4) + B(x^2 + 4) + (Cx + D)x^2;$$

so,

$$4x^3 + 6x^2 + 4x + 8 = (A + C)x^3 + (B + D)x^2 + 4Ax + 4B.$$

Equate the corresponding coefficients to obtain $B = 2$, $A = 1$, $B + D = 6$ (hence $D = 4$), and $A + C = 4$ (so $C = 3$). We check that

$$\frac{1}{x} + \frac{2}{x^2} + \frac{3x+4}{x^2+4} = \frac{x(x^2+4) + 2(x^2+4) + (3x+4)x^2}{x^2(x^2+4)} = \frac{4x^3 + 6x^2 + 4x + 8}{x^2(x^2+4)}.$$

The original integral is equal to

$$\int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{3x+4}{x^2+4} \right) dx = \boxed{\ln|x| - \frac{2}{x} + \frac{3}{2} \ln|x^2+4| + 2 \arctan \frac{x}{2} + C}$$