Math 142, Exam 2, Fall 2010

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. For each partition P of $[a, b]$ of the form $a = x_0 \le x_1 \le \cdots \le x_n = b$, let M_i be the maximum value of $f(x)$ on the subinterval $[x_{i-1}, x_i]$ and let m_i be the minimum value of $f(x)$ on $[x_{i-1}, x_i]$. If there is exactly one number with

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \text{this number} \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

as P varies over all partitions of $[a, b]$, then this number is called *the definite integral of* f *on* $[a, b]$ and this number is denoted $\int_a^b f(x)dx$.

2. (6 points) **Find** $\int_0^3 \frac{dx}{x-1}$.

This integral is improper. The function $\frac{1}{x-1}$ is not defined at $x = 1$. We see that

$$\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1^-} \int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-} \ln|x-1|\Big|_0^b = \lim_{b\to 1^-} \ln|b-1| - \ln|0-1|$$

$$= -\infty.$$

The integral $\int_0^1 \frac{dx}{x-1}$ diverges; thus the integral $\boxed{\int_0^3 \frac{dx}{x-1} \text{ also diverges.}}$

3. (6 points) **Find** $\int e^x \cos x dx$. **Check your answer.**

Let $u = e^x$ and $dv = \cos x dx$. Calculate $du = e^x dx$ and $v = \sin x$. Integration by parts gives

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

Let $u = e^x$ and $dv = \sin x dx$. Calculate $du = e^x dx$ and $v = -\cos x$. Integration by parts gives

$$\int e^x \cos x dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x dx\right).$$

Add $\int e^x \cos x dx$ to both sides to obtain

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C.$$

Divide by 2 to conclude

$$\boxed{\int e^x \cos x dx = (1/2)\left(e^x \sin x + e^x \cos x\right) + C.}$$

Check. The derivative of the proposed answer is

$$(1/2)\left[e^x \cos x + e^x \sin x - e^x \sin x + e^x \cos x\right) = e^x \cos x \checkmark.$$

4. (6 points) **Find** $\int \frac{1}{(x-2)(x^2+4)} dx$. **Check your answer.**

Use the technique of partial fractions to write

$$\frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}.$$

Multiply both sides by $(x-2)(x^2+4)$ to obtain

$$1 = A(x^2+4) + (Bx+C)(x-2)$$

$$1 = x^2(A+B) + x(C-2B) + 4A - 2C$$

Equate the corresponding coefficients to write

$$0 = A + B, \quad 0 = C - 2B, \quad 1 = 4A - 2C$$

$$A = -B, \quad 2B = C, \quad 1 = 4(-B) - 2(2B)$$

Thus, $B = -1/8$, $A = 1/8$, and $C = -2/8$. We check this much.

$$(1/8)\left[\frac{1}{x-2} + \frac{-x-2}{x^2+4}\right] = (1/8)[\frac{x^2+4-(x+2)(x-2)}{(x-2)(x^2+4)}$$

$$= (1/8)[\frac{x^2+4-(x^2-4)}{(x-2)(x^2+4)} = \frac{1}{(x-2)(x^2+4)}.$$

So the original integral is equal to

$$(1/8)\int \left[\frac{1}{x-2} + \frac{-x-2}{x^2+4}\right] dx$$

$$= \boxed{(1/8)\left[\ln|x-2| - (1/2)\ln(x^2+4) - \arctan(x/2)\right] + C}.$$

Check . The derivative of the proposed answer is

$$(1/8)\left[\frac{1}{x-2} - \frac{x}{x^2+4} - \frac{1/2}{1+(x/2)^2}\right]$$

$$= (1/8)\left[\frac{1}{x-2} - \frac{x}{x^2+4} - \frac{2}{4+x^2}\right]. \checkmark$$

5. (6 points) **Find the limit of the sequence whose n^{th} term is $a_n = (1+\frac{2}{n})^n$.**

We know that $\lim_{n\to infty}(1+\frac{2}{n})^n = \boxed{e^2}$. Of course, if we did not know this, then we would compute

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n\ln(1+\frac{2}{n}) = \lim_{n\to\infty} \frac{\ln(1+\frac{2}{n})}{\frac{1}{n}}.$$

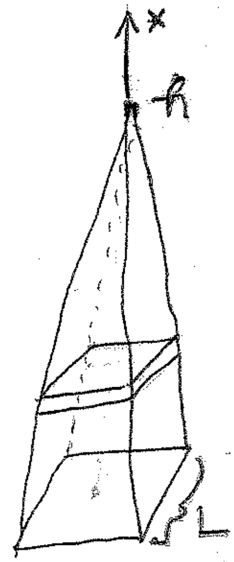
The top and the bottom both go to 0 . L'Hopital's rule shows that

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\frac{\frac{-2}{n^2}}{1+\frac{2}{n}}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}} = 2.$$

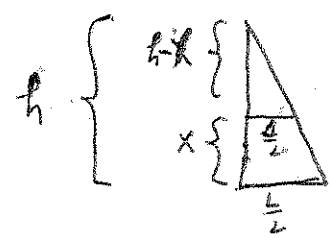
So $\lim_{n\to infty} a_n = \lim_{n\to infty} e^{\ln a_n} = e^2$.

6. (6 points) **Find the volume of the pyramid whose base is a square with side L and whose height is h .**

Consider the following picture:



Slice the pyramid into a bunch of rectangular slabs by chopping the x-axis from $x = 0$ to $x = h$. The rectangular slab with x-coordinate x has a square face of side s and a thickness dx. This rectangular slab has volume $s^2 dx$. We must relate s to x by way of h and L. The picture:



shows that

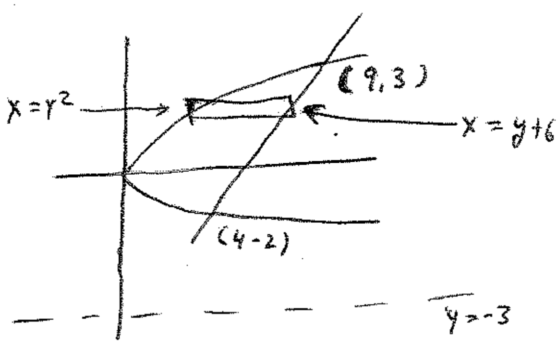
$$\frac{s/2}{h-x} = \frac{L/2}{h}$$

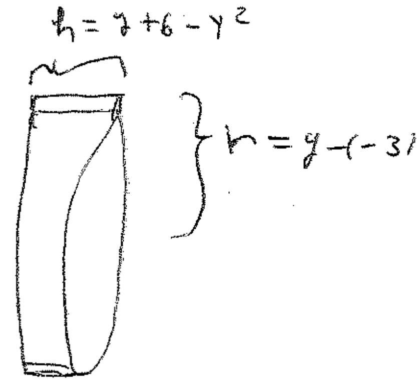
or $s = \frac{L}{h}(h-x)$. (By the way this makes sense. When $x = 0$, $s = h$ and when $x = h$, then $s = 0$.) So the volume is

$$\frac{L^2}{h^2}\int_0^h (h-x)^2 dx = \frac{L^2}{h^2}\frac{-(h-x)^3}{3}\Big|_0^h = \frac{L^2}{h^2}\frac{h^3}{3} = \boxed{\frac{L^2 h}{3}}.$$

7. (7 points) **Consider the region bounded by $x = y^2$ and $y = x - 6$. Revolve the region about $y = -3$. Find the volume of the resulting solid.**

Find the intersection points. Solve $y = y^2 - 6$, or $0 = y^2 - y - 6$, or $0 = (y - 3)(y + 2)$. So, $y = 3$ or -2 and the intersection points are $(9, 3)$ and $(4, -2)$.





Spin the rectangle. Get a shell of volume $2\pi rht$ with $t = dy$, $r = y - (-3)$, and $h = (y + 6) - y^2$. The volume of the solid is

$$2\pi \int_{-2}^{3} (y+3)(y+6-y^2)dy$$

$$= 2\pi \int_{-2}^{3} (y^2 + 6y - y^3 + 3y + 18 - 3y^2)dy$$

$$= 2\pi \int_{-2}^{3} (-2y^2 + 9y - y^3 + 18)dy$$

$$= \boxed{2\pi \left[\frac{-2y^3}{3} + \frac{9y^2}{2} - \frac{y^4}{4} + 18y\right]\Bigg|_{-2}^{3}}.$$

8. (7 points) **Find $\int \sin^3 x \cos^4 x dx$. Check your answer.**

Save $\sin x$. Convert $\sin^2 x$ into $1 - \cos^2 x$. The integral is equal to $\int \sin x(1 - \cos^2 x)\cos^4 x dx$. Let $u = \cos x$. It follows that $du = -\sin x dx$. The integral is

$$-\int (1-u^2)u^4 du = -\int (u^4 - u^6)du = -\frac{u^5}{5} + \frac{u^7}{7} + C = \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}.$$

Check . The derivative of the proposed answer is

$$-\cos^4 x(-\sin x) + \cos^6 x(-\sin x) = \sin x \cos^4 x(1 - \cos^2 x). \checkmark$$