

**Math 142, Exam 1, Spring 2011 Solutions**

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. **SHOW** your work. *CIRCLE* your answer. **CHECK** your answer whenever possible.

**No Calculators or Cell phones.**

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let  $f(x)$  be a function defined on the closed interval  $a \leq x \leq b$ . For each partition  $P$  of the closed interval  $[a, b]$  (so,  $P$  is  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ ), let  $\Delta_i = x_i - x_{i-1}$ , and pick  $x_i^* \in [x_{i-1}, x_i]$ . The definite integral  $\int_a^b f(x)dx$  is the limit over all partitions  $P$  as all  $\Delta_i$  go to zero of  $\sum_{i=1}^n f(x_i^*)\Delta_i$ .

2. (6 points) **State the Fundamental Theorem of Calculus. Be sure to explain all of your notation. (The Fundamental Theorem of Calculus as done in class has exactly one part.)**

Let  $f$  be a continuous function defined on the closed interval  $[a, b]$ . If  $F(x)$  is any antiderivative of  $f(x)$ , then  $\int_a^b f(x)dx = F(b) - F(a)$ .

3. (6 points) **Find  $\int_1^2 x\sqrt{x-1} dx$ .**

Let  $u = x - 1$ ; so  $u + 1 = x$  and  $du = dx$ . When  $x = 1$ , then  $u = 0$ . When  $x = 2$ , then  $u = 1$ . The integral is equal to

$$\int_0^1 (u + 1)\sqrt{u}du = \int_0^1 (u^{3/2} + u^{1/2})du = \frac{2}{5}(u^{5/2}) + \frac{2}{3}u^{3/2} \Big|_0^1 = \frac{2}{5} + \frac{2}{3} = \boxed{\frac{16}{15}}.$$

4. (7 points) **Find  $\int \frac{(\ln x)^2}{x} dx$ . Check your answer.**

Let  $u = \ln x$ . It follows that  $du = \frac{1}{x}dx$ . So the original integral is equal to

$$\int u^2 du = \frac{u^3}{3} + C = \boxed{\frac{(\ln x)^3}{3} + C}.$$

**Check.** The derivative of the proposed answer is

$$3 \frac{(\ln x)^2}{3} \frac{1}{x} \checkmark$$

5. (6 points) **Find**  $\int (\ln x)^2 dx$ . **Check your answer.**

Try integration by parts:  $\int u dv = uv - \int v du$ . Let  $u = (\ln x)^2$  and  $dv = dx$ . It follows that  $du = 2 \ln x \left(\frac{1}{x}\right) dx$  and  $v = x$ . The original integral is

$$x(\ln x)^2 - 2 \int \ln x dx.$$

Use integration by parts again. Let  $u = \ln x$  and  $dv = dx$ . It follows that  $du = \left(\frac{1}{x}\right) dx$  and  $v = x$ . The original integral is

$$x(\ln x)^2 - 2(x \ln x - \int dx) = \boxed{x(\ln x)^2 - 2(x \ln x - x) + C}.$$

**Check .** The derivative of the proposed answer is

$$\begin{aligned} x 2 \ln x \left(\frac{1}{x}\right) + (\ln x)^2 - 2 \left(x \left(\frac{1}{x}\right) + \ln x - 1\right) &= 2 \ln x + (\ln x)^2 - 2(1 + \ln x - 1) \\ &= (\ln x)^2. \checkmark \end{aligned}$$

6. (7 points) **Find**  $\int \sin^3 x \cos^2 x dx$ . **Check your answer.**

There is an odd power of  $\sin x$ ; so, we save one  $\sin x$  and convert everything else to  $\cos x$ . The integral is

$$\int (1 - \cos^2 x) \cos^2 x \sin x dx.$$

Let  $u = \cos x$ . It follows that  $du = -\sin x dx$ . This integral is

$$\begin{aligned} - \int (1 - u^2) u^2 du &= - \int (u^2 - u^4) du = - \left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C \\ &= \boxed{- \left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right) + C} \end{aligned}$$

**Check.** The derivative of the proposed answer is

$$\begin{aligned} - (\cos^2 x (-\sin x) - \cos^4 x (-\sin x)) &= -\cos^2 x (-\sin x) (1 - \cos^2 x) \\ &= \cos^2 x (\sin x) \sin^2 x. \checkmark \end{aligned}$$

7. (6 points) Find  $\int \sqrt{5 + 4x - x^2} dx$ . **Check your answer.**

Complete the square  $5 + 4x - x^2 = 5 + \boxed{4} - (x^2 - 4x + \boxed{4}) = 9 - (x - 2)^2$ . We let  $x - 2 = 3 \sin \theta$ . It follows that  $dx = 3 \cos \theta d\theta$  and  $9 - (x - 2)^2 = 9 - 9 \sin^2 \theta = 9 \cos^2 \theta$ . The original problem is

$$\begin{aligned} \int \sqrt{5 + 4x - x^2} dx &= \int \sqrt{9 - (x - 2)^2} dx = 9 \int \cos^2 \theta d\theta = \frac{9}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{9}{2} (\theta + (1/2) \sin 2\theta) + C = \frac{9}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{9}{2} \left( \arcsin \left( \frac{x - 2}{3} \right) + \frac{x - 2}{3} \frac{\sqrt{9 - (x - 2)^2}}{3} \right) + C \\ &= \frac{9}{2} \left( \arcsin \left( \frac{x - 2}{3} \right) + \frac{x - 2}{3} \frac{\sqrt{5 + 4x - x^2}}{3} \right) + C \\ &= \boxed{\frac{9}{2} \arcsin \left( \frac{x - 2}{3} \right) + \frac{1}{2} (x - 2) \sqrt{5 + 4x - x^2} + C} \end{aligned}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} &\frac{9}{2} \frac{1/3}{\sqrt{1 - \left(\frac{x-2}{3}\right)^2}} + (1/2) \left[ (x - 2) \frac{4 - 2x}{2\sqrt{5 + 4x - x^2}} + \sqrt{5 + 4x - x^2} \right] \\ &= \frac{9}{2} \frac{1/3}{\frac{1}{3} \sqrt{9 - (x - 2)^2}} + (1/2) \left[ (x - 2) \frac{2 - x}{\sqrt{5 + 4x - x^2}} + \sqrt{5 + 4x - x^2} \right] \\ &= \frac{9}{2} \frac{1}{\sqrt{5 + 4x - x^2}} + (1/2) \left[ (x - 2) \frac{2 - x}{\sqrt{5 + 4x - x^2}} + \sqrt{5 + 4x - x^2} \right] \\ &= \frac{1}{2\sqrt{5 + 4x - x^2}} [9 - (x - 2)^2 + 5 + 4x - x^2] \\ &= \frac{1}{2\sqrt{5 + 4x - x^2}} [2(5 + 4x - x^2)] = \sqrt{5 + 4x - x^2}. \checkmark \end{aligned}$$

8. (6 points) Find  $\int \sqrt{x^2 + 2x} dx$ . **Check your answer.**

We complete the square:  $x^2 + 2x = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1$ . Let  $x + 1 = \sec \theta$ . It follows that  $(x + 1)^2 - 1 = \tan^2 \theta$  and  $dx = \sec \theta \tan \theta d\theta$ . The original problem is equal to

$$\int \tan^2 \theta \sec \theta d\theta.$$

We use integration by parts. Let  $u = \tan \theta$  and  $dv = \sec \theta \tan \theta d\theta$ . It follows that  $du = \sec^2 \theta d\theta$  and  $v = \sec \theta$ . So

$$\begin{aligned} \int \tan^2 \theta \sec \theta d\theta &= \sec \theta \tan \theta - \int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int (\tan^2 \theta + 1) \sec \theta d\theta \\ &= \sec \theta \tan \theta - \int \sec \theta d\theta - \int \tan^2 \theta \sec \theta d\theta. \end{aligned}$$

Add  $\int \tan^2 \theta \sec \theta d\theta$  to both sides to see that

$$2 \int \tan^2 \theta \sec \theta d\theta = \sec \theta \tan \theta - \int \sec \theta d\theta.$$

So

$$\begin{aligned} \int \sqrt{x^2 + 2x} dx &= \int \tan^2 \theta \sec \theta d\theta = (1/2) \left[ \sec \theta \tan \theta - \int \sec \theta d\theta \right] \\ &= (1/2) [\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|] + C \\ &= \boxed{(1/2) \left[ (x + 1)\sqrt{x^2 + 2x} - \ln |(x + 1) + \sqrt{x^2 + 2x}| \right] + C}. \end{aligned}$$

Check. The derivative of

$$(1/2) \left[ (x + 1)\sqrt{x^2 + 2x} - \ln[(x + 1) + \sqrt{x^2 + 2x}] \right]$$

is

$$\begin{aligned} (1/2) &\left[ \frac{(x + 1)(2x + 2)}{2\sqrt{x^2 + 2x}} + \sqrt{x^2 + 2x} - \frac{1 + \frac{2x+2}{2\sqrt{x^2+2x}}}{(x + 1) + \sqrt{x^2 + 2x}} \right] \\ &= (1/2) \left[ \frac{(x + 1)^2}{\sqrt{x^2 + 2x}} + \sqrt{x^2 + 2x} - \frac{1 + \frac{x+1}{\sqrt{x^2+2x}}}{(x + 1) + \sqrt{x^2 + 2x}} \right] \end{aligned}$$

$$\begin{aligned}
&= (1/2) \left[ \frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{\sqrt{x^2+2x} + x + 1}{[(x+1) + \sqrt{x^2+2x}]\sqrt{x^2+2x}} \right] \\
&= (1/2) \left[ \frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1}{\sqrt{x^2+2x}} \right] \\
&= \frac{1}{2\sqrt{x^2+2x}} [(x+1)^2 + x^2 + 2x - 1] \\
&= \frac{1}{2\sqrt{x^2+2x}} [2x^2 + 4x] \\
&= \frac{1}{\sqrt{x^2+2x}} [x^2 + 2x] = \sqrt{x^2+2x}. \checkmark
\end{aligned}$$