Math 142, Exam 1, Spring 2011 Solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a,b]$ (so, P is $a = x_0 \le x_1 \le \cdots \le x_n = b$), let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum\limits_{i=1}^{n} f(x_i^*)\Delta_i$.

2. (6 points) **State the Fundamental Theorem of Calculus. Be sure to explain all of your notation. (The Fundamental Theorem of Calculus as done in class has exactly one part.)**

Let f be a continuous function defined on the closed interval $[a,b]$. If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. (6 points) **Find** $\int_1^2 x\sqrt{x-1}\,dx$ **.**

Let $u = x - 1$; so $u + 1 = x$ and $du = dx$. When $x = 1$, then $u = 0$. When $x = 2$, then $u = 1$. The integral is equal to

$$\int_0^1 (u+1)\sqrt{u}du = \int_0^1 (u^{3/2} + u^{1/2})du = \frac{2}{5}(u^{5/2}) + \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{5} + \frac{2}{3} = \boxed{\frac{16}{15}}.$$

4. (7 points) **Find** $\int \frac{(\ln x)^2}{x}\,dx$ **. Check your answer.**

Let $u = \ln x$. It follows that $du = \frac{1}{x}dx$. So the original integral is equal to

$$\int u^2 du = \frac{u^3}{3} + C = \boxed{\frac{(\ln x)^3}{3} + C.}$$

Check. The derivative of the proposed answer is

$$3\frac{(\ln x)^2}{3}\frac{1}{x} \checkmark$$

5. (6 points) **Find** $\int (\ln x)^2\, dx$ **. Check your answer.**

Try integration by parts: $\int u dv = uv - \int v du$. Let $u = (\ln x)^2$ and $dv = dx$. It follows that $du = 2\ln x(\frac{1}{x})dx$ and $v = x$. The original integral is

$$x(\ln x)^2 - 2\int \ln x dx.$$

Use integration by parts again. Let $u = \ln x$ and $dv = dx$. It follows that $du = (\frac{1}{x})dx$ and $v = x$. The original integral is

$$x(\ln x)^2 - 2(x\ln x - \int dx) = \boxed{x(\ln x)^2 - 2(x\ln x - x) + C}.$$

Check . The derivative of the proposed answer is

$$x2\ln x\left(\frac{1}{x}\right) + (\ln x)^2 - 2\left(x\left(\frac{1}{x}\right) + \ln x - 1\right) = 2\ln x + (\ln x)^2 - 2\left(1 + \ln x - 1\right)$$
$$= (\ln x)^2. \checkmark$$

6. (7 points) **Find** $\int \sin^3 x \cos^2 x\, dx$ **. Check your answer.**

There is an odd power of $\sin x$; so, we save one $\sin x$ and convert everything else to $\cos x$. The integral is

$$\int (1 - \cos^2 x)\cos^2 x \sin x\, dx.$$

Let $u = \cos x$. It follows that $du = \sin x dx$. This integral is

$$-\int (1-u^2)u^2 du = -\int (u^2 - u^4)du = -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$
$$= \boxed{-\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right) + C}$$

Check. The derivative of the proposed answer is

$$-\left(\cos^2 x(-\sin x) - \cos^4 x(-\sin x)\right) = -\cos^2 x(-\sin x)\left(1 - \cos^2 x\right)$$
$$= \cos^2 x(\sin x)\sin^2 x. \checkmark$$

7. (6 points) Find $\int \sqrt{5+4x-x^2}\,dx$. **Check your answer.**

Complete the square $5+4x-x^2 = 5+\boxed{4}-(x^2-4x+\boxed{4}) = 9-(x-2)^2$. We let $x-2 = 3\sin\theta$. It follows that $dx = 3\cos\theta d\theta$ and $9-(x-2)^2 = 9-9\sin^2\theta = 9\cos^2\theta$. The original problem is

$$\int \sqrt{5+4x-x^2}dx = \int \sqrt{9-(x-2)^2}dx = 9\int \cos^2\theta d\theta = \frac{9}{2}\int (1+\cos 2\theta)d\theta$$

$$= \frac{9}{2}(\theta + (1/2)\sin 2\theta) + C = \frac{9}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\,\frac{\sqrt{9-(x-2)^2}}{3}\right) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\,\frac{\sqrt{5+4x-x^2}}{3}\right) + C$$

$$= \boxed{\frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C}$$

<u>Check</u> . The derivative of the proposed answer is

$$\frac{9}{2}\frac{1/3}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}} + (1/2)\left[(x-2)\frac{4-2x}{2\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{9}{2}\frac{1/3}{\frac{1}{3}\sqrt{9-(x-2)^2}} + (1/2)\left[(x-2)\frac{2-x}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{9}{2}\frac{1}{\sqrt{5+4x-x^2}} + (1/2)\left[(x-2)\frac{2-x}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{1}{2\sqrt{5+4x-x^2}}\left[9-(x-2)^2+5+4x-x^2\right]$$

$$= \frac{1}{2\sqrt{5+4x-x^2}}\left[2(5+4x-x^2)\right] = \sqrt{5+4x-x^2}. \checkmark$$

8. (6 points) Find $\int \sqrt{x^2+2x}\,dx$. **Check your answer.**

We complete the square: $x^2+2x = (x^2+2x+1)-1 = (x+1)^2-1$. Let $x+1=\sec\theta$. It follows that $(x+1)^2-1=\tan^2\theta$ and $dx=\sec\theta\tan\theta d\theta$. The original problem is equal to

$$\int \tan^2\theta\sec\theta d\theta.$$

We use integration by parts. Let $u=\tan\theta$ and $dv=\sec\theta\tan\theta d\theta$. It follows that $du=\sec^2\theta d\theta$ and $v=\sec\theta$. So

$$\int \tan^2\theta\sec\theta d\theta = \sec\theta\tan\theta - \int \sec^3\theta d\theta = \sec\theta\tan\theta - \int(\tan^2\theta+1)\sec\theta d\theta$$

$$= \sec\theta\tan\theta - \int\sec\theta d\theta - \int\tan^2\theta\sec\theta d\theta.$$

Add $\int\tan^2\theta\sec\theta d\theta$ to both sides to see that

$$2\int\tan^2\theta\sec\theta d\theta = \sec\theta\tan\theta - \int\sec\theta d\theta.$$

So

$$\int\sqrt{x^2+2x}dx = \int\tan^2\theta\sec\theta d\theta = (1/2)\left[\sec\theta\tan\theta - \int\sec\theta d\theta\right]$$

$$= (1/2)\left[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right]+C$$

$$= \boxed{(1/2)\left[(x+1)\sqrt{x^2+2x} - \ln|(x+1)+\sqrt{x^2+2x}|\right]+C}.$$

<u>Check</u> . The derivative of

$$(1/2)\left[(x+1)\sqrt{x^2+2x} - \ln[(x+1)+\sqrt{x^2+2x}\right]$$

is

$$(1/2)\left[\frac{(x+1)(2x+2)}{2\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1+\frac{2x+2}{2\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1+\frac{x+1}{\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{\sqrt{x^2+2x}+x+1}{[(x+1)+\sqrt{x^2+2x}]\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1}{\sqrt{x^2+2x}}\right]$$

$$= \frac{1}{2\sqrt{x^2+2x}}\left[(x+1)^2 + x^2 + 2x - 1\right]$$

$$= \frac{1}{2\sqrt{x^2+2x}}\left[2x^2 + 4x\right]$$

$$= \frac{1}{\sqrt{x^2+2x}}\left[x^2 + 2x\right] = \sqrt{x^2+2x}. \checkmark$$