Math 142, Exam 1, Spring 2014

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Your work must be coherent, complete, and correct. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let $f(x)$ be a function defined on the closed interval $a \le x \le b$. For each partition P of the closed interval $[a, b]$ of the form $a = x_0 \le x_1 \le \cdots \le x_n = b$, let $\Delta_i = x_i - x_{i-1}$, and pick $x_i^* \in [x_{i-1}, x_i]$. The definite integral $\int_a^b f(x)dx$ is the limit over all partitions P as all Δ_i go to zero of $\sum\limits_{i=1}^{n} f(x_i^*)\Delta_i$.

2. **Find $\int \sin^5 x \cos^6 x dx$. Check your answer.**

Save one $\sin x$. Convert the remaining $\sin^4 x = (\sin^2 x)^2$ into cosines. Let $u = \cos x$. It follows that $du = -\sin x dx$. We see that

$$\int \sin^5 x \cos^6 x dx = \int (1 - \cos^2 x)^2 \cos^6 x \sin x dx = -\int (1 - u^2)^2 u^6 du$$

$$= -\int (1 - 2u^2 + u^4)u^6 du = -\int (u^6 - 2u^8 + u^{10})du = -(\tfrac{u^7}{7} - 2\tfrac{u^9}{9} + \tfrac{u^{11}}{11}) + C$$

$$= \boxed{-(\tfrac{\cos^7 x}{7} - 2\tfrac{\cos^9 x}{9} + \tfrac{\cos^{11} x}{11}) + C}.$$

CHECK: The derivative of the proposed answer is

$$-(\cos^6 x(-\sin x) - 2\cos^8 x(-\sin x) + \cos^{10} x(-\sin x))$$

$$= \sin x \cos^6 x(1 - 2\cos^2 x + \cos^4 x) = \sin x \cos^6 x(1 - \cos^2 x)^2$$

$$= \sin x \cos^6 x(\sin^2 x)^2 = \sin^5 x \cos^6 x. \checkmark$$

3. **Find** $\int \frac{dx}{\sqrt{12x-4x^2-8}}$ **. Check your answer.**

We see that

$$\int \frac{dx}{\sqrt{12x-4x^2-8}} = \int \frac{dx}{\sqrt{-4(x^2-3x+\boxed{\frac{9}{4}})+4\boxed{\frac{9}{4}}-8}}$$

$$= \int \frac{dx}{\sqrt{1-4(x-\frac{3}{2})^2}} = \int \frac{dx}{\sqrt{1-(2x-3)^2}}.$$

Let $u = 2x-3$. It follows that $du = 2dx$. Thus,

$$\int \frac{dx}{\sqrt{12x-4x^2-8}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C = \boxed{\frac{1}{2}\arcsin(2x-3)+C}.$$

CHECK: The derivative of the proposed answer is

$$\frac{1}{2}\frac{2}{\sqrt{1-(2x-3)^2}} = \frac{1}{\sqrt{-4x^2+12x-8}}. \checkmark.$$

4. **Find** $\int \frac{(x^2+3)dx}{(x+1)^3}$ **. Check your answer.**

We apply the technique of partial fractions and find constants A, B, and C with

$$\frac{(x^2+3)}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Multiply both sides by $(x+1)^3$ to obtain

$$x^2+3 = A(x+1)^2 + B(x+1) + C.$$

Thus,

$$\begin{aligned} x^2+3 = & Ax^2 + 2Ax + A \\ & + Bx + B \\ & + C. \end{aligned}$$

Equate the corresponding coefficients to see $A = 1$, $2A+B = 0$, and $A+B+C = 3$. It follows that $A = 1$, $B = -2$, and $C = 4$. We verify

$$\frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{4}{(x+1)^3} = \frac{(x+1)^2 - 2(x+1) + 4}{(x+1)^3} = \frac{x^2+2x+1-2x-2+4}{(x+1)^3}$$

$$= \frac{x^2+3}{(x+1)^3}.$$

Now we do the integral

$$\int \frac{(x^2+3)dx}{(x+1)^3} = \int \frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{4}{(x+1)^3} dx$$

$$= \boxed{\ln|x+1| + \frac{2}{x+1} + \frac{-2}{(x+1)^2} + C}$$

5. **Find** $\int e^{5x} \sin x dx$ **. Check your answer.**

Use integration by parts. Let $u = e^{5x}$ and $dv = \sin x dx$. Compute $du = 5e^{5x} dx$ and $v = -\cos x$. We have

$$\int e^{5x} \sin x dx = -e^{5x} \cos x + 5 \int e^{5x} \cos x dx.$$

Use intgration by parts again. Let $u = e^{5x}$ and $dv = \cos x dx$. Compute $du = 5e^{5x} dx$ and $v = \sin x$. We have

$$\int e^{5x} \sin x dx = -e^{5x} \cos x + 5 \left[e^{5x} \sin x - 5 \int e^{5x} \sin x dx \right].$$

Add $25 \int e^{5x} \sin x dx$ to both sides to see that

$$26 \int e^{5x} \sin x dx = -e^{5x} \cos x + 5e^{5x} \sin x + C.$$

Divide both sides by 26 to conclude that

$$\boxed{\int e^{5x} \sin x dx = \frac{1}{26}[-e^{5x} \cos x + 5e^{5x} \sin x] + K,}$$

where K is the constant $\frac{C}{26}$.

CHECK: The derivative of the proposed answer is

$$\frac{1}{26} \begin{bmatrix} e^{5x} \sin x - 5e^{5x} \cos x \\ +25e^{5x} \sin x + 5e^{5x} \cos x \end{bmatrix} = e^{5x} \sin x \ \checkmark.$$