

**Math 142, Exam 1, Spring 2011**

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

There are 9 problems. The exam is worth 50 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible.

**No Calculators or Cell phones.**

1. (5 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

Let  $f(x)$  be a function defined on the closed interval  $a \leq x \leq b$ . For each partition  $P$  of the closed interval  $[a, b]$  (so,  $P$  is  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ ), let  $\Delta_i = x_i - x_{i-1}$ , and pick  $x_i^* \in [x_{i-1}, x_i]$ . The definite integral  $\int_a^b f(x)dx$  is the limit over all partitions  $P$  as all  $\Delta_i$  go to zero of  $\sum_{i=1}^n f(x_i^*)\Delta_i$ .

2. (5 points) **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation. Write in complete sentences.**

Let  $f$  be a continuous function defined on the closed interval  $[a, b]$ .

(a) If  $A(x)$  is the function  $A(x) = \int_a^x f(t)dt$ , for all  $x \in [a, b]$ , then  $A'(x) = f(x)$  for all  $x \in [a, b]$ .

(b) If  $F(x)$  is any antiderivative of  $f(x)$ , then  $\int_a^b f(x)dx = F(b) - F(a)$ .

3. (5 points) **Find  $\int \sqrt{9 - 4x^2} dx$ . Check your answer.**

Let  $2x = 3 \sin \theta$ . Calculate  $2dx = 3 \cos \theta d\theta$ . Calculate

$$\sqrt{9 - 4x^2} = \sqrt{9 - 9 \sin^2 \theta} = 3\sqrt{1 - \sin^2 \theta} = 3\sqrt{\cos^2 \theta} = 3 \cos \theta.$$

The original integral is equal to

$$\frac{9}{2} \int \cos^2 \theta d\theta = \frac{9}{4} \int ((1 + \cos(2\theta)))d\theta = \frac{9}{4}(\theta + \frac{1}{2} \sin(2\theta)) + C$$

$$= \frac{9}{4}(\theta + \sin \theta \cos \theta) + C = \frac{9}{4}(\arcsin(\frac{2x}{3}) + \frac{2x}{3} \frac{\sqrt{9-4x^2}}{3}) + C$$

$$= \frac{9}{4} \arcsin(\frac{2x}{3}) + \frac{x\sqrt{9-4x^2}}{2} + C$$

Check. The derivative of the proposed answer is

$$\begin{aligned}
 & \frac{9}{4} \frac{\frac{2}{3}}{\sqrt{1 - (\frac{2x}{3})^2}} + \frac{1}{2} \left[ x \frac{-8x}{2\sqrt{9 - 4x^2}} + \sqrt{9 - 4x^2} \right] \\
 &= \frac{3}{2\sqrt{(\frac{1}{9})(9 - 4x^2)}} + \frac{-4x^2}{2\sqrt{9 - 4x^2}} + \frac{1}{2}\sqrt{9 - 4x^2} \\
 &= \frac{9}{2\sqrt{9 - 4x^2}} + \frac{-4x^2}{2\sqrt{9 - 4x^2}} + \frac{1}{2}\sqrt{9 - 4x^2} \\
 &= \frac{1}{2} \frac{9 - 4x^2}{\sqrt{9 - 4x^2}} + \frac{1}{2}\sqrt{9 - 4x^2} \\
 &= \frac{1}{2} \left[ \sqrt{9 - 4x^2} + \sqrt{9 - 4x^2} \right] = \sqrt{9 - 4x^2}. \checkmark
 \end{aligned}$$

4. (5 points) **Find**  $\int \cos 3x \cos 4x \, dx$ .

You know that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \text{and} \quad \cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Add these identities to get

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B;$$

which is the same as

$$\frac{1}{2} [\cos(A+B) + \cos(A-B)] = \cos A \cos B.$$

Our integral is

$$\frac{1}{2} \int (\cos 7x + \cos x) \, dx = \boxed{\frac{1}{2} \left( \frac{1}{7} \sin(7x) + \sin x \right) + C}.$$

5. (6 points) **Find**  $\int e^{3x} \sin x \, dx$ . **Check your answer.**

Use integration by parts. Let  $u = e^{3x}$  and  $dv = \sin x \, dx$ . It follows that  $du = 3e^{3x} \, dx$  and  $v = -\cos x$ . We have:

$$\int e^{3x} \sin x \, dx = -e^{3x} \cos x + 3 \int e^{3x} \cos x \, dx.$$

Let  $u = e^{3x}$  and  $dv = \cos x dx$ . It follows that  $du = 3e^{3x} dx$  and  $v = \sin x$ . We have:

$$\begin{aligned} \int e^{3x} \sin x dx &= -e^{3x} \cos x + 3 \int e^{3x} \cos x dx \\ &= -e^{3x} \cos x + 3 \left( e^{3x} \sin x - 3 \int e^{3x} \sin x dx \right). \end{aligned}$$

So,

$$\int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x - 9 \int e^{3x} \sin x dx.$$

Add  $9 \int e^{3x} \sin x dx$  to both sides of the equation to get

$$10 \int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x.$$

Divide by 10; don't forget to add  $+C$ . We have

$$\boxed{\int e^{3x} \sin x dx = \frac{1}{10} (-e^{3x} \cos x + 3e^{3x} \sin x) + C}.$$

Check. The derivative of the proposed answer is

$$\frac{1}{10} (-e^{3x}(-\sin x) - 3e^{3x} \cos x + 3e^{3x} \cos x + 9e^{3x} \sin x) = e^{3x} \sin x. \checkmark$$

6. (6 points) **Find**  $\int \sin^7 x dx$ . **Check your answer.**

Save one  $\sin x$ . Convert the rest to  $\cos x$ . Let  $u = \cos x$ . It follows that  $du = -\sin x$ . Our integral equals

$$\begin{aligned} \int (1 - \cos^2 x)^3 \sin x dx &= - \int (1 - u^2)^3 du = - \int (1 - 3u^2 + 3u^4 - u^6) du \\ &= -(u - u^3 + \frac{3}{5}u^5 - \frac{u^7}{7}) + C \\ &= \boxed{-(\cos x - \cos^3 x + \frac{3}{5} \cos^5 x - \frac{\cos^7 x}{7}) + C}. \end{aligned}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} -(-\sin x - 3 \cos^2 x(-\sin x) + 3 \cos^4 x(-\sin x) - \cos^6 x(-\sin x)) \\ = \sin x(1 - 3 \cos^2 x + 3 \cos^4 x - \cos^6 x) = \sin x(1 - \cos^2 x)^3. \checkmark \end{aligned}$$

7. (6 points) **Find**  $\int \tan^7 x \, dx$ . **Check your answer.**

Save  $\tan x$ , convert the rest of the tangents to secants. Our integral is equal to

$$\begin{aligned} \int (\sec^2 x - 1)^3 \tan x \, dx &= \int (\sec^6 x - 3 \sec^4 x + 3 \sec^2 x - 1) \tan x \, dx \\ &= \int (\sec^6 x - 3 \sec^4 x + 3 \sec^2 x) \tan x \, dx + \int \frac{-\sin x}{\cos x} \, dx. \end{aligned}$$

In the first integral, let  $u = \sec x$ . It follows that  $du = \sec x \tan x \, dx$ . Our integral is equal to

$$\begin{aligned} \int (u^5 - 3u^3 + 3u) \, du + \ln |\cos x| + C &= \frac{u^6}{6} - \frac{3u^4}{4} + \frac{3u^2}{2} + \ln |\cos x| + C \\ &= \boxed{\frac{\sec^6 x}{6} - \frac{3 \sec^4 x}{4} + \frac{3 \sec^2 x}{2} + \ln |\cos x| + C}. \end{aligned}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} \sec^5 x (\sec x \tan x) - 3 \sec^3 x (\sec x \tan x) + 3 \sec x (\sec x \tan x) - \tan x \\ = \tan x (\sec^6 x - 3 \sec^4 x + 3 \sec^2 x - 1) = \tan x (\sec^2 x - 1)^3 \checkmark. \end{aligned}$$

8. (6 points) **Find**  $\int \cos^2 x \, dx$ .

Our integral is  $\frac{1}{2} \int (1 + \cos 2x) \, dx = \boxed{\frac{1}{2} \left( x + \frac{\sin 2x}{2} \right) + C}$ .

9. (6 points) **Find**  $\int \sin x \sqrt{\cos^2 x + 16} \, dx$ . **Check your answer.**

Let  $w = \cos x$ . It follows that  $dw = -\sin x \, dx$ . Our integral is  $-\int \sqrt{w^2 + 16} \, dw$ .  
Let  $w = 4 \tan \theta$ . It follows that

$$dw = 4 \sec^2 \theta \, d\theta \quad \text{and} \quad \sqrt{w^2 + 16} = \sqrt{16 \tan^2 \theta + 16} = 4 \sqrt{\sec^2 \theta} = 4 \sec \theta.$$

Our integral is  $-16 \int \sec^3 \theta \, d\theta$ . We integrate  $\sec^3 \theta$  off on the side. Let  $u = \sec \theta$  and  $dv = \sec^2 \theta \, d\theta$ . It follows that

$$du = \sec \theta \tan \theta \, d\theta \quad \text{and} \quad v = \tan \theta;$$

hence

$$\int \sec^3 \theta \, d\theta = \sec \theta \tan \theta - \int \sec \theta \tan^2 \theta \, d\theta = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) \, d\theta.$$

Add  $\int \sec^3 \theta d\theta$  to both sides and divide by 2 to see that

$$\int \sec^3 \theta d\theta = \frac{1}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|] + C.$$

Our integral is

$$\begin{aligned} &= -8 [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|] + C \\ &= -8 \left[ \frac{\sqrt{w^2 + 16} w}{4} + \ln \left| \frac{\sqrt{w^2 + 16}}{4} + \frac{w}{4} \right| \right] + C \\ &= -8 \left[ \frac{\sqrt{\cos^2 x + 16} \cos x}{4} + \ln \left| \frac{\sqrt{\cos^2 x + 16}}{4} + \frac{\cos x}{4} \right| \right] + C \\ &= \frac{-\cos x \sqrt{\cos^2 x + 16}}{2} - 8 \left( \ln \left| \sqrt{\cos^2 x + 16} + \cos x \right| - \ln 4 \right) + C \end{aligned}$$

Let  $K$  be the constant  $8 \ln 4 + C$ . Our answer is

$$= \boxed{\frac{-\cos x \sqrt{\cos^2 x + 16}}{2} - 8 \ln \left| \sqrt{\cos^2 x + 16} + \cos x \right| + K}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} &-\frac{1}{2} \left( \frac{\cos x (-2 \cos x \sin x)}{2\sqrt{\cos^2 x + 16}} - \sin x \sqrt{\cos^2 x + 16} \right) - 8 \frac{\frac{-2 \cos x \sin x}{2\sqrt{\cos^2 x + 16}} - \sin x}{\sqrt{\cos^2 x + 16} + \cos x} \\ &= -\frac{1}{2} (-\sin x) \left( \frac{\cos^2 x}{\sqrt{\cos^2 x + 16}} + \sqrt{\cos^2 x + 16} \right) - 8 (-\sin x) \frac{\frac{\cos x}{\sqrt{\cos^2 x + 16}} + 1}{\sqrt{\cos^2 x + 16} + \cos x} \\ &= \frac{1}{2} \sin x \left( \frac{\cos^2 x}{\sqrt{\cos^2 x + 16}} + \sqrt{\cos^2 x + 16} \right) + 8 \sin x \frac{\cos x + \sqrt{\cos^2 x + 16}}{(\sqrt{\cos^2 x + 16} + \cos x) \sqrt{\cos^2 x + 16}} \\ &= \frac{\sin x}{2\sqrt{\cos^2 x + 16}} (\cos^2 x + \cos^2 x + 16 + 16) = \frac{\sin x}{\sqrt{\cos^2 x + 16}} (\cos^2 x + 16) = \sin x \sqrt{\cos^2 x + 16}. \checkmark \end{aligned}$$