

**Math 142, Exam 1, SOLUTION, Fall 2011**

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. **SHOW** your work. *CIRCLE* your answer. **CHECK** your answer whenever possible.

**No Calculators or Cell phones.**

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation. Write in complete sentences.**

**Answer:** Let  $f(x)$  be a function defined on the closed interval  $a \leq x \leq b$ . For each partition  $P$  of the closed interval  $[a, b]$  (so,  $P$  is  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ ), let  $\Delta_i = x_i - x_{i-1}$ , and pick  $x_i^* \in [x_{i-1}, x_i]$ . The definite integral  $\int_a^b f(x)dx$  is the limit over all partitions  $P$  as all  $\Delta_i$  go to zero of  $\sum_{i=1}^n f(x_i^*)\Delta_i$ .

2. (6 points) **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation. Write in complete sentences.**

**Answer:** Let  $f$  be a continuous function defined on the closed interval  $[a, b]$ .

(a) If  $A(x)$  is the function  $A(x) = \int_a^x f(t)dt$ , for all  $x \in [a, b]$ , then  $A'(x) = f(x)$  for all  $x \in [a, b]$ .

(b) If  $F(x)$  is any antiderivative of  $f(x)$ , then  $\int_a^b f(x)dx = F(b) - F(a)$ .

3. (6 points) **Find  $\int \sin^2 x dx$ .**

**Answer:** The given integral is equal to

$$\frac{1}{2} \int (1 - \cos 2x) dx = \boxed{\frac{1}{2} \left( x - \frac{\sin 2x}{2} \right) + C}.$$

4. (6 points) **Find  $\int \sin^5 x dx$ . Check your answer.**

**Answer:** Save one  $\sin x$  and convert the rest of the  $\sin x$ 's to  $\cos x$ . The given integral is equal to  $\int (1 - \cos^2 x)^2 \sin x dx$ . Let  $u = \cos x$ . It follows that  $du = -\sin x dx$ . The original integral is equal to

$$-\int (1 - u^2)^2 du = -\int (1 - 2u^2 + u^4) du = -(u - \frac{2}{3}u^3 + \frac{1}{5}u^5) + C$$

$$= \boxed{-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C}$$

Check. The derivative of the proposed answer is

$$\begin{aligned} \sin x + 2 \cos^2 x(-\sin x) - \cos^4 x(-\sin x) &= \sin x(1 - 2 \cos^2 x + \cos^4 x) \\ &= \sin x(1 - \cos^2 x)^2 = \sin^5 x. \checkmark \end{aligned}$$

5. (6 points) **Find**  $\int e^{3x} \sin x dx$ . **Check your answer.**

**Answer:** Use integration by parts. Let  $u = e^{3x}$  and  $dv = \sin x dx$ . It follows that  $du = 3e^{3x} dx$  and  $v = -\cos x$ . We have:

$$\int e^{3x} \sin x dx = -e^{3x} \cos x + 3 \int e^{3x} \cos x dx.$$

Let  $u = e^{3x}$  and  $dv = \cos x dx$ . It follows that  $du = 3e^{3x} dx$  and  $v = \sin x$ . We have:

$$\begin{aligned} \int e^{3x} \sin x dx &= -e^{3x} \cos x + 3 \int e^{3x} \cos x dx \\ &= -e^{3x} \cos x + 3 \left( e^{3x} \sin x - 3 \int e^{3x} \sin x dx \right). \end{aligned}$$

So,

$$\int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x - 9 \int e^{3x} \sin x dx.$$

Add  $9 \int e^{3x} \sin x dx$  to both sides of the equation to get

$$10 \int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x.$$

Divide by 10; don't forget to add  $+C$ . We have

$$\boxed{\int e^{3x} \sin x dx = \frac{1}{10} (-e^{3x} \cos x + 3e^{3x} \sin x) + C}.$$

Check. The derivative of the proposed answer is

$$\frac{1}{10} (-e^{3x}(-\sin x) - 3e^{3x} \cos x + 3e^{3x} \cos x + 9e^{3x} \sin x) = e^{3x} \sin x. \checkmark$$

6. (5 points) **Find**  $\int \frac{dx}{x^2-5x+6}$ . **Check your answer.**

**Answer:** We see that  $x^2 - 5x + 6 = (x - 2)(x - 3)$ . We use the technique of partial fractions:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

Multiply both sides by  $(x-2)(x-3)$ :

$$1 = A(x-3) + B(x-2).$$

Plug in  $x = 3$  to learn that  $B = 1$ . Plug in  $x = 2$  to learn that  $A = -1$ . Verify that

$$\frac{-1}{x-2} + \frac{1}{x-3} = \frac{-(x-3) + (x-2)}{(x-2)(x-3)} = \frac{1}{(x-2)(x-3)}.$$

The original integral is

$$\int \left( \frac{-1}{x-2} + \frac{1}{x-3} \right) dx = \boxed{-\ln|x-2| + \ln|x-3| + C.}$$

Check. The derivative of the proposed answer is

$$\frac{-1}{x-2} + \frac{1}{x-3} = \frac{1}{(x-2)(x-3)}. \checkmark$$

7. (5 points) **Find**  $\int \frac{dx}{4x^2+4x+10}$ . **Check your answer.**

**Answer:** The denominator does not factor; so we see that

$$4x^2 + 4x + 10 = (2x + 1)^2 + 9.$$

Let  $2x + 1 = 3 \tan \theta$ . We compute

$$4x^2 + 4x + 10 = (2x + 1)^2 + 9 = (3 \tan \theta)^2 + 9 = 9(\tan^2 \theta + 1) = 9 \sec^2 \theta.$$

We also compute  $2dx = 3 \sec^2 \theta d\theta$ . The integral is

$$\frac{1}{2} \int \frac{3 \sec^2 \theta d\theta}{9 \sec^2 \theta} = \frac{1}{6} \int 1 d\theta = \frac{1}{6} \theta + C = \boxed{\frac{1}{6} \arctan \left( \frac{2x+1}{3} \right) + C}$$

Check. The derivative of the proposed answer is

$$\left(\frac{1}{6}\right) \left(\frac{\frac{2}{3}}{\left(\frac{2x+1}{3}\right)^2 + 1}\right) = \frac{1}{9 \left[\left(\frac{2x+1}{3}\right)^2 + 1\right]} = \frac{1}{4x^2 + 4x + 1 + 9}. \checkmark$$

8. (5 points) **Find**  $\int \frac{e^{1/x}}{x^2} dx$ . **Check your answer.**

**Answer:** Let  $u = \frac{1}{x}$ . It follows that  $du = \frac{-dx}{x^2}$ . The integral is

$$-\int e^u du = -e^u + C = \boxed{-e^{\frac{1}{x}}}.$$

Check. The derivative of the proposed answer is

$$-e^{\frac{1}{x}} \left(\frac{-1}{x^2}\right). \checkmark$$

9. (5 points) **Find**  $\int \sqrt{1-x^2} dx$ . **Check your answer.** Let  $x = \sin \theta$ . It follows that  $dx = \cos \theta d\theta$ . One computes that

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = \cos x;$$

hence, the integral is

$$\begin{aligned} \int \cos^2 \theta d\theta &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta = \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2}\right) + C \\ &= \frac{1}{2} \left(\theta + \frac{2 \sin \theta \cos \theta}{2}\right) + C = \frac{1}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \boxed{\frac{1}{2} (\arcsin x + x \sqrt{1-x^2}) + C}. \end{aligned}$$

**Answer:**

Check. The derivative of the proposed answer is

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{\sqrt{1-x^2}} + x \frac{-2x}{2\sqrt{1-x^2}} + \sqrt{1-x^2}\right) &= \frac{1}{2\sqrt{1-x^2}} (1-x^2 + 1-x^2) = \frac{1-x^2}{\sqrt{1-x^2}} \\ &= \sqrt{1-x^2}. \checkmark \end{aligned}$$