Math 142, Exam 1, Fall 2010

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. *CIRCLE* your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (6 points) **Define the definite integral. Give a complete definition. Be sure to explain all of your notation.**

Let $f(x)$ be a continuous function defined on the closed interval $[a,b]$. For each partition P of $[a,b]$ of the form $a = x_0 \le x_1 \le \cdots \le x_n = b$, let M_i be the maximum value of $f(x)$ on the subinterval $[x_{i-1}, x_i]$ and let m_i be the minimum value of $f(x)$ on $[x_{i-1}, x_i]$. If there is exactly one number with

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \text{this number} \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

as P varies over all partitions of $[a,b]$, then this number is called *the definite integral of* f *on* $[a,b]$ and this number is denoted $\int_a^b f(x)dx$.

2. (6 points) **State both parts of the Fundamental Theorem of Calculus. Be sure to explain all of your notation.**

Let f be a continuous function defined on the closed interval $[a,b]$.

(a) If $A(x)$ is the function $A(x) = \int_a^x f(t)dt$, for all $x \in [a,b]$, then $A'(x) = f(x)$ for all $x \in [a,b]$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

3. (6 points) **Find** $\int_1^2 \frac{e^{1/x}}{x^2}dx$ **.**

Answer: Let $u = 1/x$. Then $du = -x^{-2}dx$. When $x = 1$, then $u = 1$. When $x = 2$, then $u = 1/2$. The integral is equal to

$$-\int_1^{1/2} e^u du = -e^u\big|_1^{1/2} = \boxed{e - \sqrt{e}}.$$

4. (6 points) **Find** $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$ **.**

Answer: Let $u = \ln x$. Then $du = \frac{1}{x}dx$. When $x = e$, then $u = 1$. When $x = e^4$, then $u = 4$. The integral is equal to

$$\int_1^4 u^{-1/2}du = 2\sqrt{u}\Big|_1^4 = 2\sqrt{4} - 2\sqrt{1} = \boxed{2}.$$

5. (7 points) **Find** $\int \sec^6 t dt$ **. Check your answer.**

Answer: Save $\sec^2 t$. Convert the remaining $\sec^4 t$ to $\tan t$'s using $\tan^2 t + 1 = \sec^2 t$. Let $u = \tan t$. It follows that $du = \sec^2 t dt$. The original problem is equal to

$$\int \left(\tan^2 t + 1\right)^2 \sec^2 t dt = \int (u^2+1)^2 du = \int (u^4 + 2u^2 + 1)du = \frac{u^5}{5} + \frac{2u^3}{3} + u + C$$

$$= \boxed{\frac{\tan^5 t}{5} + \frac{2\tan^3 t}{3} + \tan t + C}.$$

Check. The derivative of the proposed answer is

$$\tan^4 t \sec^2 t + 2\tan^2 t \sec^2 t + \sec^2 t = \sec^2 t(\tan^4 + 2\tan^2 t + 1) = \sec^2 t(\tan^2 t + 1)^2$$

$$= \sec^2 t \sec^4 t. \checkmark$$

6. (7 points) **Find** $\int \tan^5 x dx$ **. Check your answer.**

Answer: Save $\tan x$. Convert the remaining $\tan^4 x$ to $\sec x$'s using $\tan^2 x + 1 = \sec^2 x$. Let $u = \sec x$. It follows that $du = \sec x \tan x dx$. The original problem is equal to

$$\int \left(\sec^2 x - 1\right)^2 \tan x dx = \int \frac{(u^2-1)^2}{u} du = \int (u^3 - 2u + \tfrac{1}{u})du$$

$$= \frac{u^4}{4} - 2u^2 + \ln|u| + C = \boxed{\frac{\sec^4 x}{4} - \sec^2 x + \ln|\sec x| + C}.$$

Check. The derivative of the proposed answer is

$$\sec^3 x \sec x \tan x - 2\sec x \sec x \tan x + \tan x = \tan x(\sec^4 x - 2\sec^2 x + 1) =$$

$$\tan x(\sec^2 x - 1)^2 = \tan x \tan^4 x. \checkmark$$

7. (6 points) **Find** $\int \sqrt{5+4x-x^2}dx$ **. Check your answer.**

Answer: Complete the square $5+4x-x^2 = 5+\boxed{4}-(x^2-4x+\boxed{4}) = 9-(x-2)^2$. We let $x-2 = 3\sin\theta$. It follows that $dx = 3\cos\theta d\theta$ and $9-(x-2)^2 = 9-9\sin^2\theta = 9\cos^2\theta$. The original problem is

$$\int \sqrt{5+4x-x^2}dx = \int \sqrt{9-(x-2)^2}dx = 9\int \cos^2\theta d\theta = \frac{9}{2}\int (1+\cos 2\theta)d\theta$$

$$= \frac{9}{2}(\theta + (1/2)\sin 2\theta) + C = \frac{9}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\frac{\sqrt{9-(x-2)^2}}{3}\right) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x-2}{3}\right) + \frac{x-2}{3}\frac{\sqrt{5+4x-x^2}}{3}\right) + C$$

$$= \boxed{\frac{9}{2}\arcsin\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C}$$

<u>Check</u> . The derivative of the proposed answer is

$$\frac{9}{2}\frac{1/3}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}} + (1/2)\left[(x-2)\frac{4-2x}{2\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{9}{2}\frac{1/3}{\frac{1}{3}\sqrt{9-(x-2)^2}} + (1/2)\left[(x-2)\frac{2-x}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{9}{2}\frac{1}{\sqrt{5+4x-x^2}} + (1/2)\left[(x-2)\frac{2-x}{\sqrt{5+4x-x^2}} + \sqrt{5+4x-x^2}\right]$$

$$= \frac{1}{2\sqrt{5+4x-x^2}}\left[9-(x-2)^2+5+4x-x^2\right]$$

$$= \frac{1}{2\sqrt{5+4x-x^2}}\left[2(5+4x-x^2)\right] = \sqrt{5+4x-x^2}. \checkmark$$

8. (6 points) **Find** $\int \sqrt{x^2+2x}dx$ **. Check your answer.**

Answer: We complete the square: $x^2+2x=(x^2+2x+1)-1=(x+1)^2-1$. Let $x+1=\sec\theta$. It follows that $(x+1)^2-1=\tan^2\theta$ and $dx=\sec\theta\tan\theta d\theta$. The original problem is equal to

$$\int \tan^2\theta\sec\theta d\theta.$$

We use integration by parts. Let $u=\tan\theta$ and $dv=\sec\theta\tan\theta d\theta$. It follows that $du=\sec^2\theta d\theta$ and $v=\sec\theta$. So

$$\int \tan^2\theta\sec\theta d\theta=\sec\theta\tan\theta-\int\sec^3\theta d\theta=\sec\theta\tan\theta-\int(\tan^2\theta+1)\sec\theta d\theta$$

$$=\sec\theta\tan\theta-\int\sec\theta d\theta-\int\tan^2\theta\sec\theta d\theta.$$

Add $\int\tan^2\theta\sec\theta d\theta$ to both sides to see that

$$2\int\tan^2\theta\sec\theta d\theta=\sec\theta\tan\theta-\int\sec\theta d\theta.$$

So

$$\int\sqrt{x^2+2x}dx=\int\tan^2\theta\sec\theta d\theta=(1/2)\left[\sec\theta\tan\theta-\int\sec\theta d\theta\right]$$

$$=(1/2)\left[\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right]+C$$

$$=\boxed{(1/2)\left[(x+1)\sqrt{x^2+2x}-\ln\left|(x+1)+\sqrt{x^2+2x}\right|\right]+C}.$$

Check . The derivative of

$$(1/2)\left[(x+1)\sqrt{x^2+2x}-\ln[(x+1)+\sqrt{x^2+2x}\right]$$

is

$$(1/2)\left[\frac{(x+1)(2x+2)}{2\sqrt{x^2+2x}}+\sqrt{x^2+2x}-\frac{1+\frac{2x+2}{2\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}}\right]$$

$$=(1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}}+\sqrt{x^2+2x}-\frac{1+\frac{x+1}{\sqrt{x^2+2x}}}{(x+1)+\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{\sqrt{x^2+2x}+x+1}{[(x+1)+\sqrt{x^2+2x}]\sqrt{x^2+2x}}\right]$$

$$= (1/2)\left[\frac{(x+1)^2}{\sqrt{x^2+2x}} + \sqrt{x^2+2x} - \frac{1}{\sqrt{x^2+2x}}\right]$$

$$= \frac{1}{2\sqrt{x^2+2x}}\left[(x+1)^2 + x^2 + 2x - 1\right]$$

$$= \frac{1}{2\sqrt{x^2+2x}}\left[2x^2 + 4x\right]$$

$$= \frac{1}{\sqrt{x^2+2x}}\left[x^2 + 2x\right] = \sqrt{x^2+2x}. \checkmark$$