Recitation Time_____ PRINT your name ______________________________

Math 141, Exam 4, Solutions Spring 2009

The exam is worth a total of 50 points. There are 8 questions on 5 pages. **SHOW your work. Make your work be coherent and clear.** Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

I will post the solutions sometime this afternoon.

1. (7 points) **Let** $y = x\arcsin(2x)$ **. Find** $\frac{dy}{dx}$ **.**

We see that

$$\boxed{\frac{dy}{dx} = \frac{2x}{\sqrt{1-4x^2}} + \arcsin(2x).}$$

2. (7 points) **Let** $y = e^{x\sin x}$ **. Find** $\frac{dy}{dx}$ **.**

We see that

$$\boxed{\frac{dy}{dx} = (x\cos x + \sin x)e^{x\sin x}.}$$

3. (6 points) **Find** $\lim\limits_{x\to 0^+} x\ln x$ **.**

We see that

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}.$$

The top and the bottom both become infinite as x goes to 0 from above; so we apply L'Hôpital's rule to see that

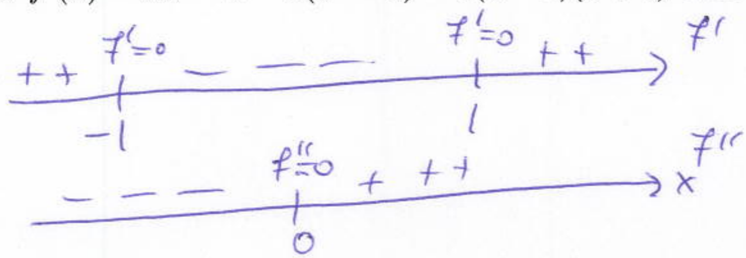
$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}.$$

Multiply top and bottom by $-x^2$ to see that

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} -x = \boxed{0}.$$

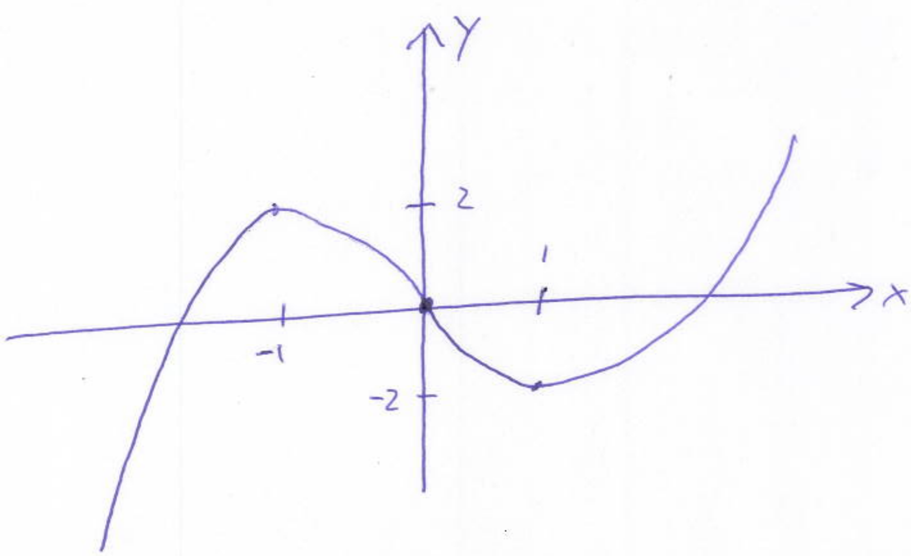
4. (6 points) **Let $f(x) = x^3 - 3x$. Where is $f(x)$ increasing and decreasing? Where is $f(x)$ concave up and concave down? Find the local extreme points and points of inflection of $y = f(x)$? Graph $y = f(x)$.**

We calculate $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$ and $f''(x) = 6x$.



We calculate $f(-1) = 2$, $f(0) = 0$, and $f(1) = -2$. We conclude that

$f(x)$ is increasing for $x < -1$ or $1 < x$, $f(x)$ is decreasing for $-1 < x < 1$, $f(x)$ is concave up for $0 < x$, $f(x)$ is concave down for $x < 0$, $(-1, 2)$ is a local maximum point of $y = f(x)$, $(1, -2)$ is a local minimum point of $y = f(x)$, and $(0, 0)$ is a point of inflection of $y = f(x)$.



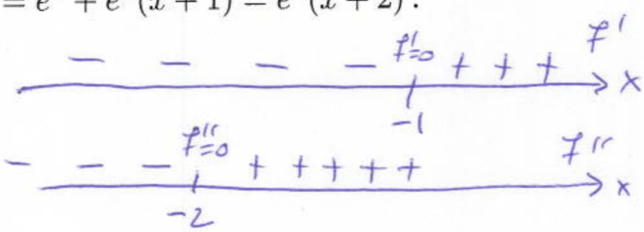
5. (6 points) **Let $f(x) = xe^x$. Find all vertical and horizontal asymptotes of $y = f(x)$. Where is $f(x)$ increasing and decreasing? Where is $f(x)$ concave up and concave down? Find the local extreme points and points of inflection of $y = f(x)$? Graph $y = f(x)$.**

The function $f(x)$ does not become infinite near any number x because this function never has zero in the demomonator and this function never trys to take ln of zero. So there are no vertical asymptotes. It is obvious that $\lim_{x\to\infty} f(x) = \infty$ so there is no horizontal asymptote on the right side of the graph. However, we may use L'Hôpital's rule (since x and e^{-x} both become infinte as x goes to $-\infty$) to see that

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^x = 0;$$

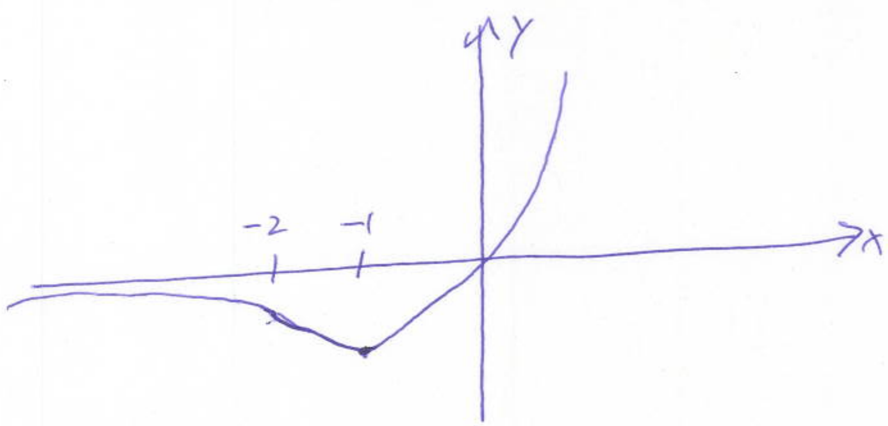
hence the x-axis is a horizontal asymptote of the graph (on the left side).

We now calculate $f'(x) = xe^x + e^x = e^x(x+1)$ and
$f''(x) = e^x + e^x(x+1) = e^x(x+2)$.



We calculate $f(-1) = -1/e$ and $f(-2) = -2/e^2$. We conclude that

$f(x)$ is increasing for $-1 < x$, $f(x)$ is decreasing for $x < -1$, $f(x)$ is concave up for $-2 < x$, $f(x)$ is concave down for $x < -2$, $(-1, -1/e)$ is a local minimum point of $y = f(x)$, $(-2, -2/e^2)$ is a point of inflection of $y = f(x)$, and $y = 0$ is a horizontal asymptote of $y = f(x)$.

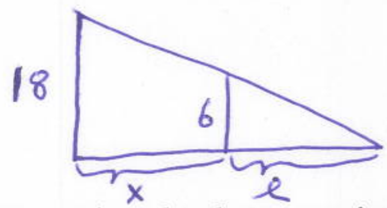


6. (6 points) **A man 6 feet tall is walking away from an 18 foot tall street light at the rate of 7 ft/sec. At what rate is his shadow lengthening? Be sure to give units.**

Let x be the distance from the man to the light pole and ℓ be the length of his shadow. We are told $\frac{dx}{dt} = 7$ ft/sec. We want to find $\frac{d\ell}{dt}$. Use similar triangles to see that

$$\frac{\ell}{6} = \frac{\ell + x}{18}.$$

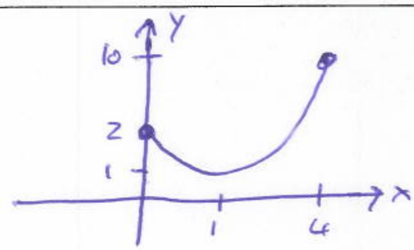
Clean the last equation up to see $3\ell = \ell + x$ or $2\ell = x$. Take the derivative to learn $2\frac{d\ell}{dt} = \frac{dx}{dt}$. We conclude that $\boxed{\frac{d\ell}{dt} = \frac{7}{2}\,\frac{\text{ft}}{\text{sec}}.}$



7. (6 points) **Find the absolute maximum points and absolute minimum points of the function $f(x) = x^2 - 2x + 2$ which is defined for $0 \le x \le 4$.**

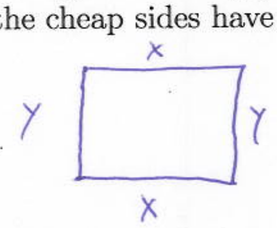
We see that $f'(x) = 2x - 2$; so $f'(x)$ is equal to zero when $x = 1$. The absolute extreme points of f occur either at the critical point $x = 1$ or at one of the endpoints $x = 0$ or $x = 4$. We calculate $f(0) = 2$, $f(1) = 1$ and $f(4) = 10$. We conclude that

$\boxed{(1,1) \text{ is the absolute minimum point and } (4,10) \text{ is the absolute maximum point.}}$

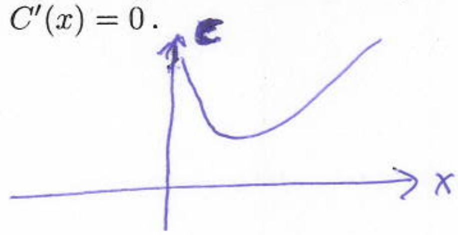


8. (6 points) **A rectangular area of 3200 square feet is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.**

Our rectangle has sides of length x and y, measured in feet. Let the expensive sides have length x and the cheap sides have length y.



We want to minimize the cost C. We see that $C = 2(2x) + 2y$ dollars. We are told that $xy = 3200$. So, we minimize $C(x) = 4x + 6400/x$, for $0 < x$. It is clear that C goes to infinity as x goes to 0 or infinity; so the minimum cost occurs at the x which causes $C'(x) = 0$.



We calculate

$$C'(x) = 4 - 6400/x^2 = \frac{4x^2 - 6400}{x^2} = \frac{4(x^2 - 1600)}{x^2} = \frac{4(x-40)(x+40)}{x^2}.$$

The only positive x with $C'(x) = 0$ is $x = 40$. The cheapest fence has sides $\boxed{40\,\text{ft} \times 80\,\text{ft}}$ and the expensive sides have length 40 feet.