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PRINT Your Name:______________________________

There are 19 problems on 9 pages. One of the problems is worth 20 points. Each of the other problems is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!** You might find the following formulas to be useful:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

1. Let $f(x) = \sqrt{3\sin^3 2x + \frac{3}{x}}$. Find $f'(x)$.

$$f'(x) = \frac{18\sin^2 2x \cos 2x - \frac{3}{x^2}}{2\sqrt{3\sin^3 2x + \frac{3}{x}}}$$

2. Compute $\int x\sqrt{x^2+1}dx$. (Be sure to check your answer.)

$$\frac{1}{2}\frac{2}{3}(x^2+1)^{\frac{3}{2}} + C$$

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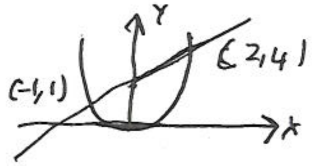
3. Find the area of the region which is bounded by $y = x^2$ and $y - x = 2$.



intersection $x^2 - x - 2 = 0$

$(x-2)(x+1)$

$x = 2, -1$

$$\text{area} = \int_{-1}^{2} (x+2-x^2)\,dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big]_{-1}^{2} = 2 + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

4. Use the DEFINITION of the DERIVATIVE to find the derivative of $f(x) = \frac{1}{3x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{3(x+h)} - \frac{1}{3x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{3h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{3\cancel{h}(x+h)x} = \lim_{h\to 0} \frac{-1}{3(x+h)x} = -\frac{1}{3x^2}$$

5. Find $\lim_{x\to 2^+} \frac{x-2}{x^2-5x+6} = \lim_{x\to 2^+} \frac{x-2}{(x-2)(x-3)} = \lim_{x\to 2^+} \frac{1}{x-3} = -1$

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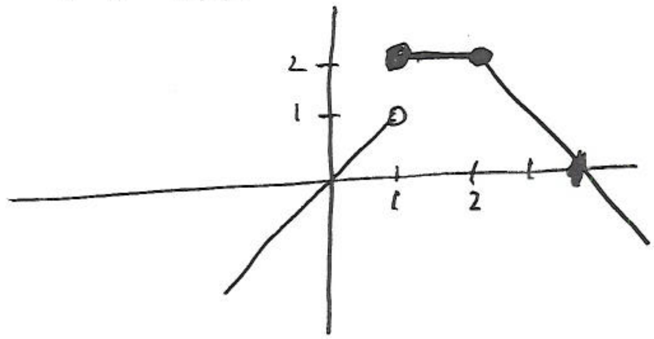
6. (20 points) Let

$$f(x) = \begin{cases} x & \text{if } x < 1, \\ 2 & \text{if } 1 \le x \le 2, \\ 4 - x & \text{if } 2 < x. \end{cases}$$

(a) Graph $y = f(x)$.



(b) Fill in the blanks:

$f(1) = $ 2	$\lim_{x\to1^+} f(x) = $ 2	$\lim_{x\to1^-} f(x) = $ 1	$\lim_{x\to1} f(x) = $ ONE
$f(2) = $ 2	$\lim_{x\to2^+} f(x) = $ 2	$\lim_{x\to2^-} f(x) = $ 2	$\lim_{x\to2} f(x) = $ 2
$f(3) = $ 1	$\lim_{x\to3^+} f(x) = $ 1	$\lim_{x\to3^-} f(x) = $ 1	$\lim_{x\to3} f(x) = $ 1

7. Find the legth of the curve $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$.

$$\int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(x^3 - \tfrac{1}{4}x^{-3}\right)^2}\, dx = \int_1^2 \sqrt{x^6 - \tfrac{1}{2} + \tfrac{1}{16}x^{-6} + 1}\, dx$$

$$= \int_1^2 \sqrt{x^6 + \tfrac{1}{2} + \tfrac{1}{16}x^{-6}}\, dx = \int_1^2 \sqrt{\left(x^3 + \tfrac{1}{4}x^{-3}\right)^2}\, dx = \int_1^2 x^3 + \tfrac{1}{4}x^{-3}\, dx$$

$$= \frac{x^4}{4} - \frac{1}{8}x^{-2}\Big]_1^2 = 4 - \frac{1}{32} - \left(\frac{1}{4} - \frac{1}{8}\right)$$

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8. Each edge of a cube is increasing at the rate of 4 inches per second. How fast is the surface area of the cube increasing when an edge is 12 inches long?

$A = 6\ell^2$

We know $\frac{d\ell}{dt} = 4$ in/sec

We want $\left.\frac{dA}{dt}\right|_{\ell=12}$

$\frac{dA}{dt} = 12\ell \frac{d\ell}{dt}$

$\left.\frac{dA}{dt}\right|_{\ell=12} = 12 \cdot 12 \cdot 4 \ \frac{in^2}{sec}$

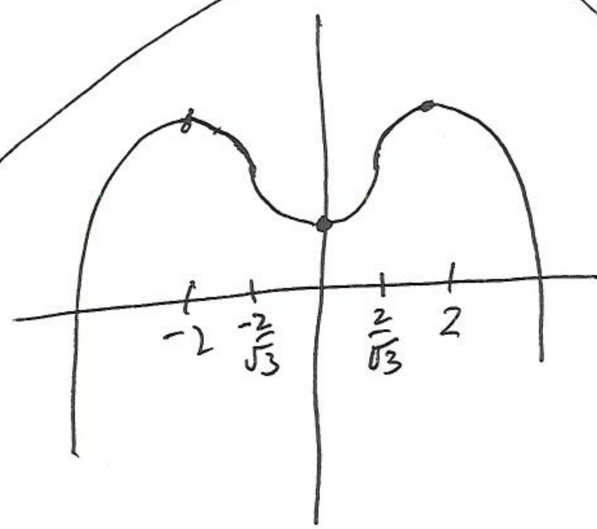
9. Let $f(x) = 1 + 8x^2 - x^4$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local maximum points, local minimum points, and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes of $y = f(x)$. Graph $y = f(x)$.

$f' = 16x - 4x^3 = 4x(4 - x^2) = 4x(2-x)(2+x)$

$f'' = 16 - 12x^2 = 4(4 - 3x^2) = 4(2 - \sqrt{3}x)(2 + \sqrt{3}x)$

f' pos | -2 | f' neg | 0 | f' pos | 2 | f' neg

f'' neg | $-\frac{2}{\sqrt{3}}$ | f'' pos | $\frac{2}{\sqrt{3}}$ | f'' neg



No V or H asy

f is inc. for $x < -2$ also for $0 < x < 2$

f is dec for $-2 < x < 0$ also for $2 < x$

f is c.u. for $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$

f is c.d. for $x < -\frac{2}{\sqrt{3}}$ also for $\frac{2}{\sqrt{3}} < x$

$(-2, f(-2))$ $(2, f(2))$ are loc. max

$(0, 1)$ is a loc. min

$\left(-\frac{2}{\sqrt{3}}, f\left(-\frac{2}{\sqrt{3}}\right)\right)$ and $\left(\frac{2}{\sqrt{3}}, f\left(\frac{2}{\sqrt{3}}\right)\right)$ are p of i

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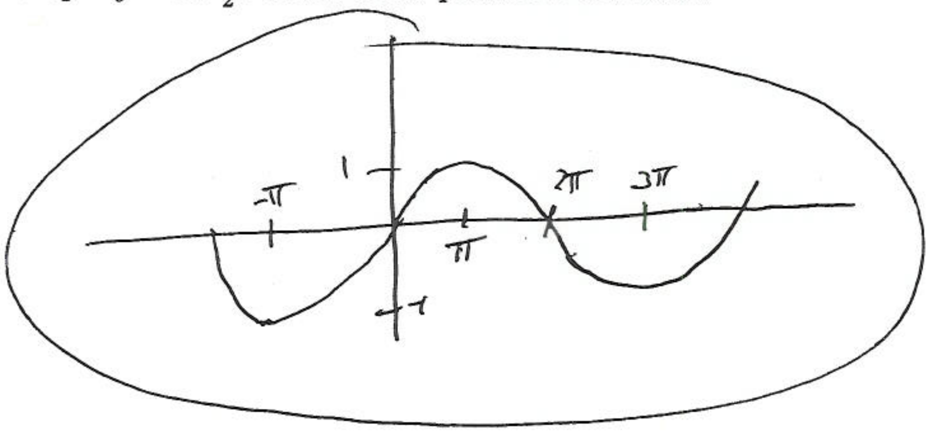
10. Graph $y = \sin \frac{x}{2}$. Mark a few points on each axis.

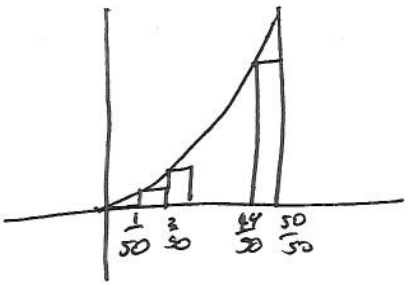


11. Find $\frac{dy}{dx}$ for $6x^4y^3 + \sin(2x^2y^3) = 19y^2$.

$$6x^4 3y^2 \frac{dy}{dx} + 24x^3y^3 + \cos(2x^2y^3)\left[2x^2 3y^2 \frac{dy}{dx} + 4xy^3\right] = 38y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-24x^3y^3 - 4xy^3\cos(2x^2y^3)}{18x^4y^2 + 4xy^3\cos(2x^2y^3) - 38y}$$

12. Consider the region A, which is bounded by the $x-$ axis, $y = x^3$, $x = 0$, and $x = 3$. Consider 150 rectangles, all with base $1/50$, which UNDER estimate the area of A. How much area is inside the 150 rectangles? (You must answer the question I asked. I expect an exact answer in closed form.)



$$\frac{1}{50}\cdot 0 + \frac{1}{50}\cdot\left(\frac{1}{50}\right)^3 + \frac{1}{50}\left(\frac{2}{50}\right)^3 + \cdots + \frac{1}{50}\left(\frac{49}{50}\right)^3$$

$$= \left(\frac{1}{50}\right)^4 \left(1^3 + 2^3 + \cdots + 49^3\right)$$

$$= \left(\frac{1}{50}\right)^4 \frac{(49)^2(50)^2}{4}$$

13. Solve the Initial Value Problem $\frac{dy}{dx} = y^3$, $y(-1) = 1$. (Be sure to check your answer.)

$$\frac{dy}{y^{-3}} = dx$$

$$\frac{y^{-2}}{-2} = X + C$$

$$\frac{1}{-2X + K} = y^2$$

$$+\frac{1}{\sqrt{K - 2x}} = y$$

$y(-1) = 1$ so our $y = +\frac{1}{\sqrt{K-2x}}$

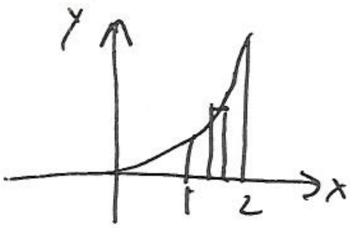
positive

$$1 = \frac{1}{\sqrt{K+2}}$$

So $K = -1$

$$y = \frac{1}{\sqrt{-1-2x}}$$

14. Let R be the region between $y = x^2$ and the $x-$ axis, from $x = 1$ to $x = 2$. Find the volume of the solid which is obtained by revolving R about the $y-$ axis.



spin the rectangle get a shell
of vol $2\pi rht$ where $t = dx$
$r = x$ $h = x^2$

$$Vol = \int_1^2 2\pi x^3 \, dx$$

$$= \frac{2\pi x^4}{4}\Big]_1^2 = \frac{\pi}{2}[16-1]$$

15. Define the definite integral $\int_a^b f(x)dx$.

Let $f(x)$ be a function which is defined for $a \le x \le b$. For each partition P of the interval from a to b (so P is $a = x_0 \le x_1 \le \cdots \le x_n = b$

Let $U_p(f) = (x_1 - x_0)M_1 + (x_2 - x_1)M_2 + \cdots + (x_n - x_{n-1})M_n$ and

$L_p(f) = (x_1 - x_0)m_1 + (x_2 - x_1)m_2 + \cdots + (x_n - x_{n-1})m_n$

where M_i is the maximum value of $f(x)$ on $x_{i-1} \le x \le x_i$

and m_i is the minimum value of $f(x)$ on $x_{i-1} \le x \le x_i$

for each i with $1 \le i \le n$. If there is exactly one number which is between every $L_p(f)$ and every $U_p(f)$ then that number is the definite integral of $f(x)$ on $a \le x \le b$ and this number is called $\int_a^b f(x) dx$.

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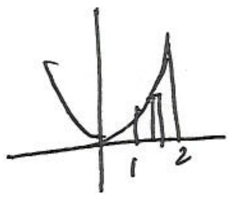
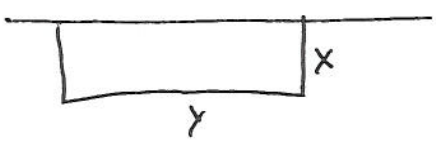
16. Let R be the region between $y = x^2$ and the $x-$ axis, from $x = 1$ to $x = 2$. Find the volume of the solid which is obtained by revolving R about the $x-$ axis.



slice the rect. get a disc of
vol $\pi r^2 t$ where $t = dx$ $r = x^2$

$$\text{Vol} = \int_1^2 \pi x^4 \, dx = \frac{\pi x^5}{5}\Big]_1^2 = \frac{\pi}{5}(32-1)$$

17. Farmer Brown has 100 feet of fence with which he plans to enclose a rectangular pen along one side of his 150-foot barn. (The side along the barn needs no fence.) What are the dimensions of the pen that has maximum area?



$2X + Y = 100$

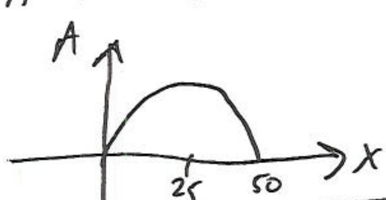
Maximize $A = XY$

$A = X(100 - 2X)$ $\quad 0 \le X \le 50$

$A = 100X - 2X^2$

$A' = 100 - 4X$

$A' = 0$ when $X = 25$

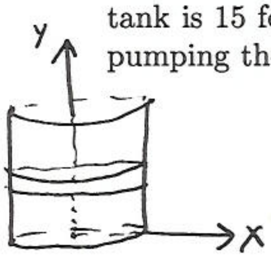


Area is maximized when the pen is 25 ft by 50 ft

18. Compute $\int \frac{x+1}{\sqrt{x}}\,dx$. (Be sure to check your answer.)

$$= \int x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

19. A tank in the shape of a right circular cylinder, standing on its end, is full of water. The density of water is 62.4 pounds per cubic foot. If the height of the tank is 15 feet and the radius of the tank is 5 feet, then find the work done in pumping the water over the top edge of the tank.



The work to raise the "slab" of water
with y-coordinate y is
$wt \cdot dist = (\pi r^2 t)\, 62.4 \cdot (15-y)$
when $r=5$ $t=dy$

$$\text{total work} = \int_0^{15} 62.4\pi\, 25\, (15-y)\,dy$$

$$= 62.4\pi\, 25 \left(15y - \frac{y^2}{2}\right)\Big]_0^{15}$$

$$= 62.4\,\pi\,(25)\,\frac{15\cdot 15}{2} \text{ ft-lbs.}$$