1999 Ex4

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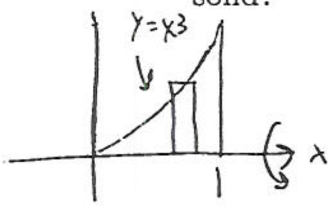
PRINT Your Name:____________________ Recitation Time ____________

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!** You might find the following formulas to be useful:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

1. Consider the region bounded by $y = x^3$, the x-axis, $x = 0$ and $x = 1$. Revolve this region about the x-axis. What is the volume of the resulting solid?



spin each rectangle get a disc of vol $\pi r^2 t$

where $r = x^3$ and $t = dx$

each disc has vol $\pi(x^3)^2 dx$ Add them up Take the limit

$Vol = \pi\int_0^1 x^6\,dx = \pi \frac{x^7}{7}\Big]_0^1 = \frac{\pi}{7}$

2. Find the area of the region between $x = y^2$ and $2y + x = 3$.

intersection

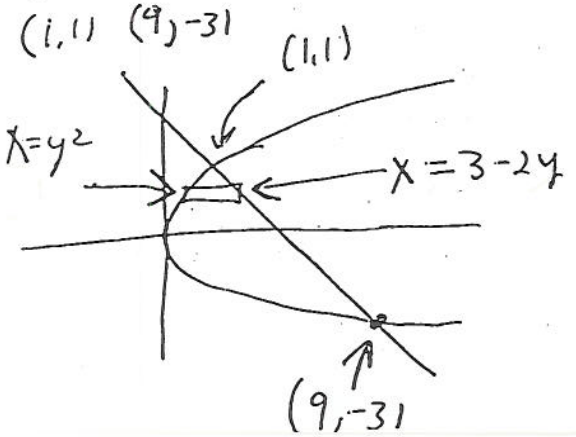
$2y + y^2 - 3 = 0$

$(y+3)(y-1) = 0$

$y = -3, 1$

intersection is

$(1,1)$ $(9,-3)$



$$\text{area} = \int_{-3}^{1} 3 - 2y - y^2 \, dy$$

$$= 3y - y^2 - \frac{y^3}{3}\Big]_{-3}^{1}$$

$$= 3 - 1 - \frac{1}{3} - (-9 - 9 + 9)$$

$$= 11 - \frac{1}{3}$$

$$= \frac{32}{3}$$

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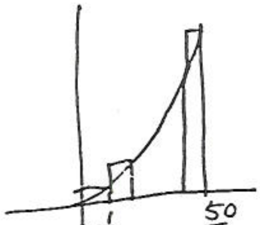
3. Consider the region bounded by $y = x^3$, the x-axis, $x = 0$ and $x = 1$. Approximate the area of this region using 50 rectangles. The rectangles have equal bases. Make your recatangles OVER estimate the area of the region. How much area is INSIDE the 50 RECTANGLES? (Be sure to answer the question that I asked.)



$$\text{area} = \frac{1}{50}\cdot\left(\frac{1}{50}\right)^3 + \frac{1}{50}\left(\frac{2}{50}\right)^3 + \cdots + \frac{1}{50}\left(\frac{50}{50}\right)^3$$

$$= \frac{1}{50^4}\left(1^3 + 2^3 + \cdots + 50^3\right)$$

$$= \frac{1}{50^4}\,\frac{50^2(51)^2}{4}$$

$$= \left(\frac{51}{50}\right)^2 \frac{1}{4}$$

4. Define the definite integral $\int_a^b f(x)dx$.

Let $f(x)$ be a function which is defined for $a \le x \le b$. For each Partition P of the interval from a to b (so P is $a = x_0 \le x_1 \le \cdots \le x_n = b$

Let $U_P(f) = (x_1 - x_0)M_1 + (x_2 - x_1)M_2 + \cdots + (x_n - x_{n-1})M_n$ and

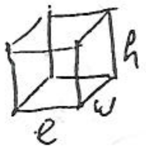
$L_P(f) = (x_1 - x_0)m_1 + (x_2 - x_1)m_2 + \cdots + (x_n - x_{n-1})m_n$

where M_i is the maximum value of $f(x)$ on $x_{i-1} \le x \le x_i$

and m_i is the minimum value of $f(x)$ on $x_{i-1} \le x \le x_i$

for each i with $1 \le i \le n$. If there is exactly one number which is between every $L_P(f)$ and every $U_P(f)$ then that number is the definite integral of $f(x)$ on $a \le x \le b$ and this number is called $\int_a^b f(x)\,dx$.

5. An open box with a capacity of 36 cubic feet is needed. If the box must be twice as long as it is wide, what dimensions would require the least amount of material?



$36 = lwh = 2w^2h$ so $\frac{18}{w^2} = h$

$l = 2w$

Area $= 2w^2 + 2wh + 4wh$

bot, l+r, f+b

$A = 2w^2 + 6wh$

$A = 2w^2 + \frac{6w}{w^2} \cdot 18$

$A = 2w^2 + \frac{6 \cdot 18}{w}$

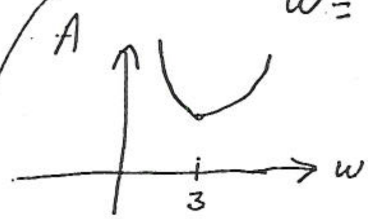
$A' = 4w - \frac{6 \cdot 18}{w^2}$

$A' = 0$ whe

$4w^3 = 6 \cdot 18$

$w^3 = \frac{6 \cdot 18}{4} = 3.9$

$w = 3$



The least amount of material is used when

$w = 3$ ft
$l = 6$ ft
$h = 2$ ft

6. Let $f(x) = x^4 - 2x^2$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

No h. asy (because no denominator)
No v. asy (because $\lim_{x \to \pm\infty} f = +\infty$)

$f' = 4x^3 - 4x$
$= 4x(x^2 - 1)$

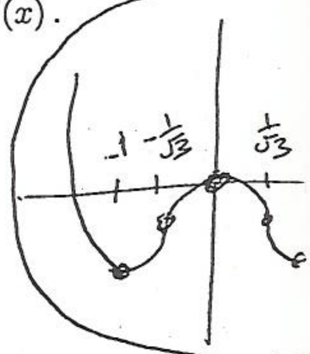
f' neg, f' pos, f' neg, f' pos
$-1 \quad 0 \quad 1$

(1, −1) and (−1, −1) are loc. min
(0,0) is a local max
f is increasing for $-1 < x < 0$ also for $1 < x$
f is dec. for $x < -1$ also for $0 < x < 1$

$f'' = 12x^2 - 4$
$= 4(3x^2 - 1)$
$= 4(\sqrt{3}x + 1)(\sqrt{3}x - 1)$

$+ \quad - \quad +$
$-\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}$

f is c.u. for $x < -\frac{1}{\sqrt{3}}$ also for $\frac{1}{\sqrt{3}} < x$
f is c.d. for $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$
$\left(\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)\right)$ and $\left(-\frac{1}{\sqrt{3}}, f\left(-\frac{1}{\sqrt{3}}\right)\right)$ are points of inflection



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7. Let $3x^4y^3 = \cos(5x^3y^6)$. Find $\frac{dy}{dx}$.

$$3x^4\,3y^2\frac{dy}{dx} + 12x^3y^3 = -\sin(5x^3y^6)\left[5x^3\,6y^5\frac{dy}{dx} + 15x^2y^6\right]$$

$$\left[9x^4y^2 + 30x^3y^5\sin(5x^3y^6)\right]\frac{dy}{dx} = -15x^2y^6\sin(5x^3y^6) - 12x^3y^3$$

$$\frac{dy}{dx} = \frac{-15x^2y^6\sin(5x^3y^6) - 12x^3y^3}{9x^4y^2 + 30x^3y^5\sin(5x^3y^6)}$$

both Parts of 6

8. State the Fundamental Theorem of Calculus.

Let $f(x)$ be a continuous function for $a \le x \le b$

1) If $A(x) = \int_a^x f(t)\,dt$, then $A'(x) = f(x)$.

2) If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - f(a).$$

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9. Solve $\frac{dy}{dx} = \frac{x}{y}$, with $y(1) = -2$. (Be sure to check your answer).

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$y^2 = x^2 + 2C$

$y = +\sqrt{x^2+2C}$ or $-\sqrt{x^2+2C}$

but y is sometimes negative so

$y = -\sqrt{x^2+2C}$

when $x=1$ then $y=-2$ so

$-2 = -\sqrt{1+2C}$

$4 = 1+2C$

$3 = 2C$

so

$y = -\sqrt{x^2+3}$

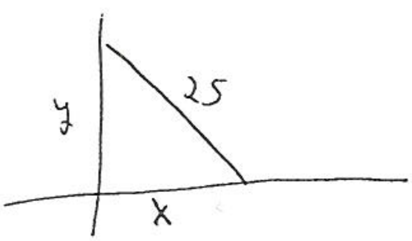
check

when $x=1$ $y=-2$ ✓

$\frac{dy}{dx} = \frac{-2x}{2\sqrt{x^2+3}} = \frac{-x}{\sqrt{x^2+3}}$

$\frac{x}{y} = \frac{x}{-\sqrt{x^2+3}}$ the same ✓

10. A 25-foot ladder is leaning against a wall. If the bottom of the ladder is pulled along the level pavement directly away from the wall at 3 feet per second, how fast is the top of the ladder moving down the wall when the foot of the ladder is 7 feet from the wall?



y = dist from top of ladder to the ground

x = dist. from bot of ladder to wall

we know $\frac{dx}{dt} = 3$ ft/s

we want $\left.\frac{dy}{dt}\right|_{x=7}$

$x^2+y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{2x}{2y}\frac{dx}{dt}$

when $x=7$ $y = \sqrt{25^2-49}$

$\left.\frac{dy}{dt}\right|_{x=7} = \frac{-7}{\sqrt{25^2-49}} \cdot 3\ \frac{ft}{s}$