1996 Exam 4

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PRINT Your Name:______________________ Section:__________

There are 10 problems on 5 pages. Each problems is worth 10 point. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS!** You might find the following formulas to be useful:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

1. State the Mean Value Theorem. If $f(x)$ is a differentiable function for $a \le x \le b$, then there exists a number c such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

2. Define the definite integral. For each partition of the interval $a \le x \le b$ as $a = x_0 \le x_1 \le \dots \le x_n = b$, let M_i be the maximum of f on $[x_{i-1}, x_i]$, m_i be the minimum of f on $[x_{i-1}, x_i]$, U be the upper sum $M_1(x_1-x_0) + M_2(x_2-x_1) + \dots + M_n(x_n-x_{n-1})$ and L be the lower sum $m_1(x_1-x_0)+m_2(x_2-x_1)+\dots+m_n(x_n-x_{n-1})$. If there is exactly one number between every upper sum and every lower sum, then that number is the definite integral $\int_a^b f(x)\,dx$.

3. Find $\int x\left(2x^2 + \frac{1}{x}\right) dx$. (Check your answer.)

$$= \int (2x^3 + 1)\,dx$$

$$= \frac{x^4}{2} + x + c$$

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4. Find $\int (\cos^4 x^3)(x^2 \sin x^3)dx$. (Check your answer.)

Let $u = \cos x^3$

$du = -3x^2 \sin x^3 \, dx$

$= \int u^4 \left(-\frac{1}{3}\right) du$

$= \frac{u^5}{-15} + C$

$= -\frac{1}{15}\cos^5 x^3 + C$

5. Find $\int x\sqrt{x+1}dx$. (Check your answer.)

Let $u = x+1$

$du = dx$

$u-1 = x$

$= \int (u-1)\sqrt{u}\, du$

$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$

$= \boxed{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$

6. Solve the Initial Value Problem $\frac{dy}{dt} = t^3y^2$, $y(2) = 1$. (Check your answer.)

*

$\int y^{-2} dy = \int t^3 dt$

$\frac{-1}{y} = \frac{t^4}{4} + C$

$\frac{-1}{\frac{t^4}{4} + C} = y$

$\frac{-1}{4+C} = 1$

$-1 = 4 + C$

$C = -5$

$y = \frac{-1}{\frac{t^4}{4} - 5}$

$\boxed{y = \frac{-4}{t^4 - 20}}$

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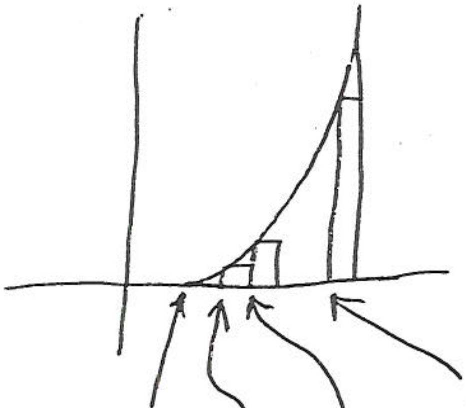
7. Consider the region A, which is bounded by the $x-$ axis, $\cancel{y = x^2 - 1}$ $y=(x-1)^2$, $x = 1$, and $x = 2$. Consider 50 rectangles, all with base $1/50$, which UNDER estimate the area of A. How much area is inside the 50 rectangles? (You must answer the question I asked. I expect an exact answer.)

in closed form



$1+\frac{0}{50}$ $1+\frac{1}{50}$ $1+\frac{2}{50}$ $1+\frac{49}{50}$

$$\text{area} = \frac{1}{50}\left(\left(1+\frac{0}{50}-1\right)^2 + \left(1+\frac{1}{50}-1\right)^2 + \cdots + \left(1+\frac{49}{50}-1\right)^2\right)$$

$$= \frac{1}{50}\left(\frac{0^2}{50^2} + \frac{1^2}{50^2} + \cdots + \frac{49^2}{50^2}\right)$$

$$= \frac{1}{50^3}\left(1^2 + 2^2 + \cdots + 49^2\right)$$

$$= \frac{1}{50^3}\,\frac{(49)50(99)}{6}$$

$$= \boxed{\frac{(49)(99)}{6(50)^2}}$$

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8. Let $f(x) = x^{5/3} - x^{2/3}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

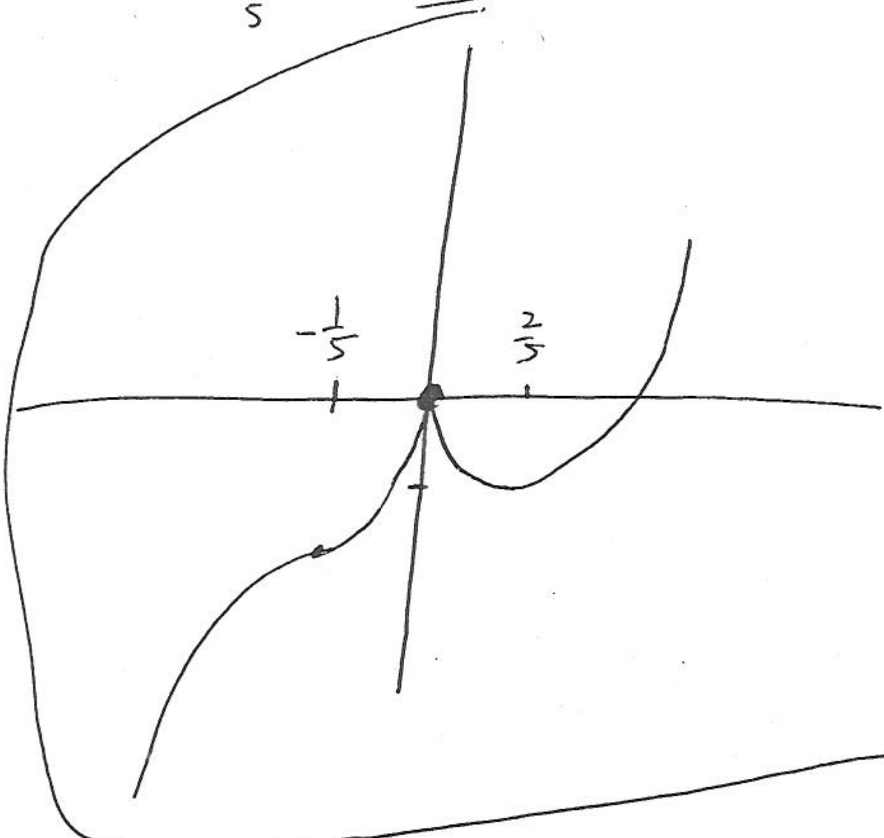
No V or h asymptotes.

$$f' = \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{-1}{3}} = \frac{1}{3}x^{\frac{-1}{3}}(5x-2)$$

$$f'' = \frac{10}{9}x^{\frac{-1}{3}} + \frac{2}{9}x^{-\frac{4}{3}} = \frac{2}{9}x^{\frac{-4}{3}}(5x+1)$$

$0 < f'$ | f' DNE (0) | $f' < 0$ | $f' = 0$ ($\frac{2}{5}$) | $0 < f'$

f'' neg | $-\frac{1}{5}$ | f'' pos | 0 | f'' pos



inc for $x<0$, also for $\frac{2}{5}<x$

dec for $0<x<\frac{2}{5}$

c.u. for $-\frac{1}{5}<x$

c.d. for $x<-\frac{1}{5}$

local max $(0,0)$

local min $(\frac{2}{5}, f(\frac{2}{5}))$

p of i at $(-\frac{1}{5}, f(-\frac{1}{5}))$

No v. or h asymptotes.

9. Find the points on the curve $y = 10 - x^2$ which are closest to the point $(0,0)$.

Let (x,y) be a point on the parabola

We must minimize $f = x^2 + y^2$

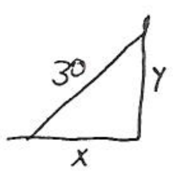
so $f = 10 - y + y^2$ with $y \le 10$

$f' = -1 + 2y$

$f' = 0$ when $y = \frac{1}{2}$ (This obviously is the min of $f(y)$ because $f(y)$ is a parabola which of the form $\cup$.)

So the closest points are $\left(\pm\sqrt{10 - \frac{1}{2}}, \frac{1}{2} \right)$

10. A 30– foot ladder is leaning against a wall. If the bottom of the ladder is pulled along the level pavement directly away from the wall at 3 feet per second, how fast is the top of the ladder moving down the wall when the foot of the ladder is 5 feet from the wall?



Know $\frac{dx}{dt} = 3\ f/s$ $\qquad x^2 + y^2 = (30)^2$

Want $\left.\frac{dy}{dt}\right|_{x=5\,f}$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

$$\boxed{\left.\frac{dy}{dt}\right|_{x=5} = \frac{-5}{\sqrt{900-25}}(3)\ f/s}$$