1995 Exam 4

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PRINT Your Name:____________________________

There are 13 problems on 7 pages. Problems 1 and 2 are each worth 6 points. Each of the other problems is worth 8 points. SHOW your work. $\boxed{CIRCLE}$ your answer. You might find the following formulas to be useful:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

NO CALCULATORS!

1. State both parts of the Fundamental Theorem of Calculus.

Let $f(x)$ be continuous for $a \le x \le b$

a) If $A(x) = \int_a^x f(t)\,dt$, then $A'(x) = f(x)$.

b) If $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

2. Define the definite integral.

Let $f(x)$ be a function which is defined for $a \le x \le b$. For each partition $P: a = x_0 \le x_1 \le \cdots \le x_n = b$, let

$U_p = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$ and

$L_p = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$,

where M_i is the maximum of $f(x)$ on $[x_{i-1}, x_i]$ and m_i is the minimum of $f(x)$ on $[x_{i-1}, x_i]$.

If there is exactly one number with

$L_p \le \# \le U_p$ for all partitions P,

then that # is called the definite integral of f on $[a,b]$ and it is denoted $\int_a^b f(x)\,dx$.

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3. Let $y = \sqrt{x\cos^3(4x^2+3)+\sin^4(x)}$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-3x\cos^2(4x^2+3)\sin(4x^2+3)\,8x + \cos^3(4x^2+3) + 4\sin^3 x\cos x}{2\sqrt{x\cos^3(4x^2+3)+\sin^4 x}}$$

4. Find $\int \frac{2}{x^2} + \sin(2x)dx$. $= \frac{-2}{x} - \frac{\cos 2x}{2} + c$

5. Find $\int \frac{\sin x\cos x}{\sqrt{2\sin^2 x+1}}dx$. $= \int \frac{1}{4} u^{-\frac{1}{2}}\,du = \frac{1}{4}\,2\,u^{\frac{1}{2}} + c$

$u = 2\sin^2 x + 1$ $\quad 4\sin x\cos x + 0$

$du = 4\sin x\cos x\,dx$

$= \frac{1}{2}(2\sin^2 x+1)^{\frac{1}{2}} + c$

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6. Let $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? What are the local extreme points and points of inflection of $y = f(x)$. Find all vertical and horizontal asymptotes. Graph $y = f(x)$.

$$f' = \frac{(x-2)(2x-2) - (x^2-2x+4)}{(x-2)^2} = \frac{2x^2-6x+4-(x^2-2x+4)}{(x-2)^2}$$

$$= \frac{x^2-4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$$

$$f'' = \frac{(x-2)^2(2x-4) - (x^2-4x)2(x-2)}{(x-2)^4} = \frac{(x-2)\left((x-2)(2x-4) - 2(x^2-4x)\right)}{(x-2)^4}$$

$$= \frac{2x^2-8x+8-2x^2+8x}{(x-2)^3} = \frac{8}{(x-2)^3}$$

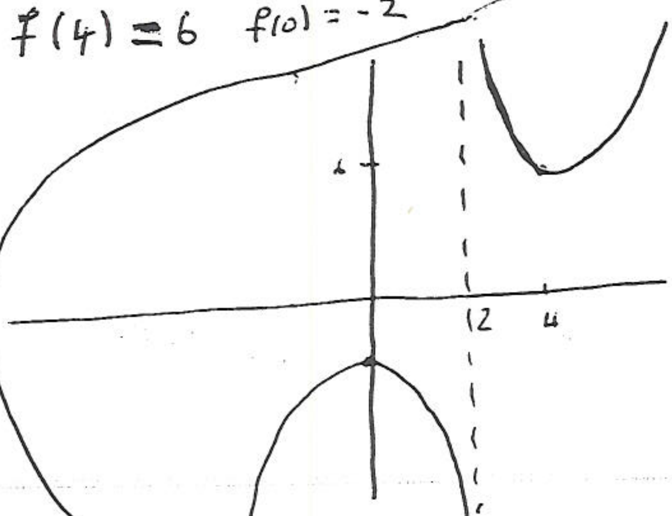
f': $+$ | 0 | $-$ | 2 | $-$ | 4 | $+$ $\to x$

f'': $-$ | 2 | $+$ $\to x$

$\lim_{x\to 2^+} f = +\infty$ $\quad \lim_{x\to\infty} f = +\infty$

$\lim_{x\to 2^-} f = -\infty$ $\quad \lim_{x\to -\infty} f = -\infty$

$f(4) = 6 \quad f(0) = -2$



V. asy is $x=2$

No h. asy

loc max $(4,6)$

loc min $(0,-2)$

No pt of inf.

inc for $4<x$, also for $x<0$

dec for $0<x<2$, also for $2<x<4$

c.u for $x<2$

cd for $x<2$

7. The surface area of a cube is growing at the constant rate of 1000 square inches per second. How fast is the volume growing when each edge is 5 inches long?

Let ℓ = length of each edge $\quad S = 6\ell^2 \quad \frac{dS}{dt} = 1000 \quad$ Find $\left.\frac{dV}{dt}\right|_{\ell=5}$

S = surface area $\quad V = \ell^3$

V = vol.

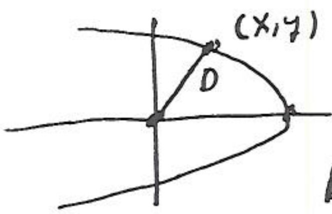
$$\frac{dS}{dt} = 12\ell\frac{d\ell}{dt} \qquad \text{So } \frac{1}{12\ell}\frac{dS}{dt} = \frac{d\ell}{dt}$$

$$\frac{dV}{dt} = 3\ell^2\frac{d\ell}{dt} = \frac{3\ell^2}{12\ell}\frac{dS}{dt} = \frac{\ell}{4}\frac{dS}{dt}$$

$$\left.\frac{dV}{dt}\right|_{\ell=5} = \frac{5}{4}1000 = 1250\ \frac{in^3}{sec}$$

8. Find the points on the curve $y^2 + 2x = 9$ which are closest to the point $(0,0)$.

$y^2 =$



Let (x,y) be a point on the curve

Let D be the distance from (x,y) to $(0,0)$.

$$D = \sqrt{x^2+y^2} = \sqrt{x^2+9-2x}$$

$$\frac{dD}{dx} = \frac{2x-2}{2\sqrt{x^2+9-2x}}$$

D' $- - -$ $+ + +$ at $x = 1$

The minimum occurs when $x=1$

the closest points are $(1,\sqrt{7})$, $(1,-\sqrt{7})$

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9. Solve the Initial Value Problem $\frac{dy}{dx} = x^3y^2$, $y(2) = 1$.

$$\int \frac{dy}{y^2} = \int \quad dx$$

$$\boxed{\frac{-4}{x^4 - 20} = y}$$

$$-\frac{1}{y} = \frac{x^4}{4} + C$$

$$-1 = 4 + C$$

$$-5 = C$$

$$\frac{-1}{\frac{x^4}{4} - 5} = y$$

10. Let $f(x) = x^2 + x$. Simplify the expression $\sum_{i=1}^{n} f(\frac{3i}{n})$. Your answer is not allowed to have a summation sign or ... .

$$\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 + \frac{3i}{n}\right] = \frac{9}{n^2}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} i = \frac{9}{n^2}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\,\frac{n(n+1)}{2}$$

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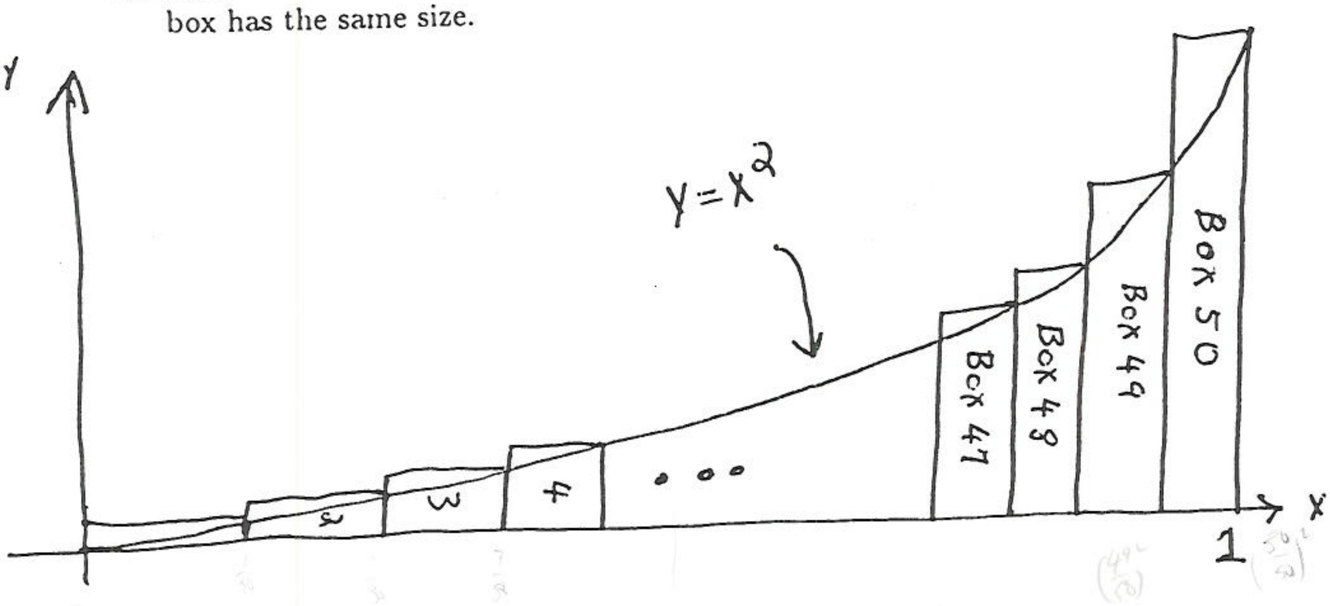
11. Find the **exact** amount of area inside the following 50 **boxes**. The base of each box has the same size.



$$\text{area} = \frac{1}{50}\cdot\frac{1}{(50)^2} + \frac{1}{50}\left(\frac{2}{50}\right)^2 + \frac{1}{50}\left(\frac{3}{50}\right)^2 + \cdots + \frac{1}{50}\left(\frac{50}{50}\right)^2$$

$$= \left(\frac{1}{50}\right)^3 (1^2 + 2^2 + 3^2 + \cdots + 50^2)$$

$$= \left(\frac{1}{50}\right)^3 \frac{(50)(51)(101)}{6}$$

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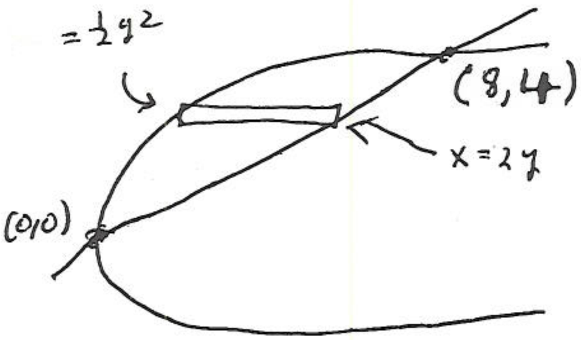
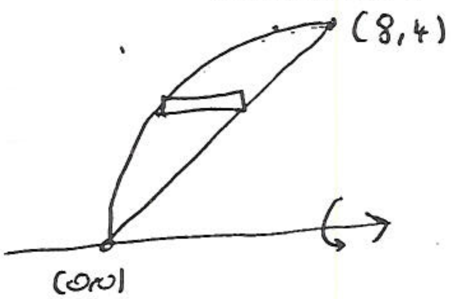
12. Find the area of region between $x - 2y = 0$ and $y^2 - 2x = 0$.



intersection:
$y^2 - 4y = 0$
$y = 0, 4$

$$\int_0^4 2y - \tfrac{1}{2}y^2\,dy = y^2 - \frac{y^3}{6}\Big]_0^4 = 16 - \frac{32}{3} = \frac{16}{3}$$

13. Find the volume of the solid which is obtained by revolving the region of problem 12 about the $x-$ axis.



spin the rectangle, get a shell of volume $2\pi r h t$
$t = dy$
$h = 2y - \frac{1}{2}y^2$
$r = y$

$$Vol = 2\pi \int_0^4 y(2y - \tfrac{1}{2}y^2)\,dy = 2\pi \int_0^4 2y^2 - \tfrac{1}{2}y^3\,dy$$

$$= 2\pi \left[\tfrac{2}{3}y^3 - \tfrac{1}{8}y^4\right]_0^4 = 2\pi \cdot 64 \left[\tfrac{2}{3} - \tfrac{1}{2}\right] = 2\pi\, 64 \left(\tfrac{1}{6}\right) = \frac{64\pi}{3}$$