Socle degrees of Frobenius powers
Lecture 2 — January 25, 2006
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The purpose of these talks is to prove (some parts of) the following result.

Theorem. *Let k be a field of positive characteristic p, P be the polynomial ring $k[x_1, \ldots, x_n]$, C be the homogeneous complete intersection ideal $C = (f_1, \ldots, f_m)$ in P and R be P/C. Let I be a homogeneous ideal in P with P/I a finite dimensional vector space over k. Suppose that the socle degrees of R/IR are $d_1 \le \cdots \le d_\ell$ and that the socle degrees of $R/I^{[p]}R$ are $D_1 \le \cdots \le D_L$. Then the following statements are equivalent:*
(1) $L = \ell$ and $D_i = pd_i - (p-1)a(R)$ for all i, and
(2) The ring R/IR has finite projective dimension as an R-module.

Remark. In the present context $a(R)$ is $\sum |f_i| - \sum |x_i|$.

Proof of (1) $\Leftarrow$ (2) **when** $C = 0$. The ring P is regular. Every P-module has a finite resolution by free P-modules. Let

$$\mathbb{F}: \quad 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$$

be the minimal homogeneous resolution of P/I by free P-modules, with $F_n = \bigoplus_i P(-b_i)$. There are two ingredients to the proof.

(A) The number of back twists in $\mathbb{F}$ is exactly equal to the dimension of the socle of P/I; furthermore, b_i and $d_i = b_i + a(P)$.

(B) One obtains the minimal free resolution of $P/I^{[p]}$ by applying the Frobenius functor to $\mathbb{F}$.

As soon as you buy (A) and (B), then the proof is complete. Ingredient (B) tells us that the back twists in the P-resolution of P/I are pb_i, with $1 \le i \le \ell$. Thus, by (A):

$$D_i = pb_i + a(P) = p(b_i + a(P)) - (p-1)a(P) = pd_i - (p-1)a(P).$$

A quick illustration. Let $P = k[x, y]$ and $I = (x^2, xy, y^2)$. The P-resolution of P/I is

$$0 \to P(-3)^2 \xrightarrow{\begin{bmatrix} x & 0 \\ y & x \\ 0 & y \end{bmatrix}} P(-2)^3 \xrightarrow{\begin{bmatrix} y^2 & -xy & x^2 \end{bmatrix}} P.$$

The resolution of $P/I^{[p]}$ is

$$0 \to P(-3p)^2 \xrightarrow{\begin{bmatrix} x^p & 0 \\ y^p & x^p \\ 0 & y^p \end{bmatrix}} P(-2p)^3 \xrightarrow{\begin{bmatrix} y^{2p} & -x^p y^p & x^{2p} \end{bmatrix}} P.$$

We have $a(P) = -2$. We saw that x and y form a basis for the socle of P/I. So the socle degrees of P/I are $d_1 = 1 \le d_2 = 1$. The back twists in the resolution of P/I are $b_1 = 3 \le b_2 = 3$. We see that $3 - 2 = 1$, so $b_i + a(P) = d_i$. We also see that $x^{p-1}y^{2p-1}$ and $x^{2p-1}y^{p-1}$ are in the socle of $P/I^{[p]}$. One can show that $x^{p-1}y^{2p-1}$ and $x^{2p-1}y^{p-1}$ are a basis for the socle of $P/I^{[p]}$. So the socle degrees of $P/I^{[p]}$ are $D_1 = 3p-2 \le D_2 = 3p-2$; the back twists in the resolution of $P/I^{[p]}$ are $B_1 = 3p \le B_2 = 3p$; and $a(P)$ is still -2. We have $D_i = B_i + a(P)$ and also $D_i = pd_i - (p-1)a(P)$, for both i.

Ingredient (B), in the present form, is due to Kunz (1969) – this is the paper that got commutative algebraists (especially Peskine, Spziro, Hochster) using Frobenius methods. One could also think of this assertion as an application of "What makes a complex exact?" (John Olmo lectured on this last Fall). The complex $\mathbb{F}$ is a resolution, so the ranks of its matrices behave correctly and the grade of the ideals of matrix minors grow correctly. If one raises each entry of each matrix to the p^{th} power, then the ranks of the new matrices are the same as the ranks of the old matrices (since $\det M^{[p]} = (\det M)^p$ because the characteristic of the ring is p), and the grade of the ideals of minors also remains unchanged!

I will give two explanations for ingredient (A). The quick argument is that one may commute $\mathrm{Tor}_n^P(P/I, k)$ using either coordinate. If one resolves P/I, then applies $\_ \otimes_P k$, and then computes homology, then one sees that

$$\mathrm{Tor}_n^P(P/I, k) = \bigoplus_i k(-b_i).$$

In other words, the generators of Tor have degrees $b_1 \le \dots$. On the other hand, if one resolves k, then applies $P/I \otimes_P \_$, and then computes homology, then one sees that

$$\mathrm{Tor}_n^P(P/I, k) = \bigoplus_i \frac{I : \mathfrak{m}}{I}(a(P)).$$

In other words, the the generators of Tor have degrees $d_1 - a(P), \dots$ (where the socle degrees of P/I are $d_1, \dots$). So $d_i = b_i + a(P)$ as claimed.

My second argument is exactly the same as my first, except, instead of stating the abstract result that Tor may computed in either coordinate, I reprove this result, giving a construction which associates an element of the socle of P/I to each basis vector at the back of the resolution of of P/I. The constructive argument takes longer, but shows what is really happening. Let $\mathbb{F}$ be a resolution of P/I, as above. Let $\mathbb{G}$ be the Koszul complex which resolves k. One can directly show that there is an isomorphism

$$(*) \qquad H_n(\mathbb{F} \otimes k) \cong H_n(\mathrm{Tot}(\mathbb{F} \otimes \mathbb{G})) \cong H_n(\tfrac{P}{I} \otimes \mathbb{G}).$$

Anyhow, I think that the best way to convey the idea of (*) is to work out the example where $I = (x^2, xy, y^2)$. In this case, $\mathbb{F}$ is

$$0 \to \underbrace{P(-3)^2}_{F_2} \xrightarrow{f_2 = \begin{bmatrix} x & 0 \\ y & x \\ 0 & y \end{bmatrix}} \underbrace{P(-2)^3}_{F_1} \xrightarrow{f_1 = \begin{bmatrix} y^2 & -xy & x^2 \end{bmatrix}} \underbrace{P}_{F_0},$$

$\mathbb{G}$ is

$$0 \to \underbrace{P(-2)}_{G_2} \xrightarrow{g_2 = \begin{bmatrix} y \\ -x \end{bmatrix}} \underbrace{P(-1)^2}_{G_1} \xrightarrow{g_1 = \begin{bmatrix} x & y \end{bmatrix}} \underbrace{P}_{G_0},$$

and $\mathbb{F}\otimes\mathbb{G}$ is

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & F_2\otimes G_2 & \xrightarrow{f_2\otimes 1} & F_1\otimes G_2 & \xrightarrow{f_1\otimes 1} & F_0\otimes G_2\\
 & & \downarrow{\scriptstyle 1\otimes g_2} & & \downarrow{\scriptstyle 1\otimes g_2} & & \downarrow{\scriptstyle 1\otimes g_2}\\
0 & \longrightarrow & F_2\otimes G_1 & \xrightarrow{f_2\otimes 1} & F_1\otimes G_1 & \xrightarrow{f_1\otimes 1} & F_0\otimes G_1\\
 & & \downarrow{\scriptstyle 1\otimes g_1} & & \downarrow{\scriptstyle 1\otimes g_1} & & \downarrow{\scriptstyle 1\otimes g_1}\\
0 & \longrightarrow & F_2\otimes G_0 & \xrightarrow{f_2\otimes 1} & F_1\otimes G_0 & \xrightarrow{f_1\otimes 1} & F_0\otimes G_0.
\end{array}$$

Start with $\begin{bmatrix}1\\0\end{bmatrix}\otimes 1$ in $F_2\otimes G_0$ in the lower left hand corner. We see that this element represents an element of the homology of $H_2(\mathbb{F}\otimes k)$. One can extend this element to get an element of the homology of $H_2(\operatorname{Tot}(\mathbb{F}\otimes\mathbb{G}))$:

$$\begin{array}{ccccc}
 & & & & 1\otimes y\\
 & & & & \downarrow\\
 & & \begin{bmatrix}1\\0\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} & \longrightarrow & 1\otimes\begin{bmatrix}y^2\\-xy\end{bmatrix}\\
 & & \downarrow & & \\
\begin{bmatrix}1\\0\end{bmatrix} & \longrightarrow & \begin{bmatrix}x\\y\\0\end{bmatrix} & &
\end{array}$$

The indicated element of $H_2(\operatorname{Tot}(\mathbb{F}\otimes\mathbb{G}))$ gives rise to the element y of the socle of P/I. To answer the question that our freshman ask: “Yes, it always works like that.” We can use the idea of the snaky game to prove both isomorphisms in (*).

Now we work on (1) $\implies$ (2). We want to prove that $\operatorname{pd}_R(R/IR)<\infty$. We apply the Theorem of Avramov and Claudia Miller (see the last seminar talk given by John Olmo last semester.) It suffices to prove that $\operatorname{Tor}_1^R(R/IR,{}^{\varphi}R)=0$. In other words, it suffices to show that if

$$(**)\qquad \begin{array}{c} R^{b_2}\xrightarrow{d_2}R^{b_1}\xrightarrow{d_1}R\to R/I\to 0\quad\text{is exact}\\ \implies R^{b_2}\xrightarrow{d_2^{[p]}}R^{b_1}\xrightarrow{d_1^{[p]}}R\to R/I^{[p]}\to 0\quad\text{is exact.}\end{array}$$

In other words, it suffices to show that

$$(***) \qquad I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C.$$

I will show that (***) implies (**). (This is a rather grubby calculation. I do it to show that our goal is very concrete! One can read the calculation backwards to show that (**) implies (***).)

I make my calculation at the P-level. Let $a_1, \dots, a_{b_1}$ generate I in P; so,

$$d_1 = [\, a_1 \quad \dots \quad a_{b_1} \,]$$

and

$$d_1^{[p]} = [\, a_1^p \quad \dots \quad a_{b_1}^p \,].$$

We think of d_2 as having two pieces:

$$d_2 = [\, d_2' \quad d_2'' \,]$$

where

$$P^{b_2{}'} \xrightarrow{d_2'} P^{b_1} \xrightarrow{d_1} P$$

is exact (and d_2'' is all of the extra columns that describe elements of I which are also in C.) Recall that Kunz's Theorem (ingredient (B) of the other direction) tells us that

$$P^{b_2{}'} \xrightarrow{(d_2')^{[p]}} P^{b_1} \xrightarrow{d_1^{[p]}} P$$

is exact.

Suppose v is in P^{b_1} with $d_1^{[p]}(v) \in C$. In other words,

$$d_1^{[p]}(v) \in I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C.$$

So, there exist $s_1, \dots, s_t \in I \cap C$; $\alpha_1, \dots, \alpha_t$ in P; and $c_1, \dots c_{b_1}$ in C so that

$$d_1^{[p]}(v) = \sum_{i=1}^{t} \alpha_i s_i^p + \sum_{i=1}^{b_1} a_i^p c_i.$$

Of course, there exists $v_i \in P^{b_1}$ with $d_1(v_i) = s_i$ (and therefore also $d_1^{[p]} v_1^{[p]} = s_i^p$). So,

$$d_1^{[p]}(v) = d_1^{[p]} \left(\sum_{i=1}^{t} \alpha_i v_i^{[p]} + \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix} \right).$$

So,

$$v - \sum_{i=1}^{t} \alpha_i v_i^{[p]} - \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix}$$

is killed by $d_1^{[p]}$; hence is in the image of $(d_2')^{[p]}$. Finally, $d_1(v_i) = s_i \in I \cap C$, so $v_i = d_2''(w_i)$ for some w_i; hence, $v_i^{[p]} = (d_2'')^{[p]}(w_i^{[p]})$. Thus,

$$v \in \operatorname{im} d_2^{[p]} + CP^{b_1},$$

as desired.

To be continued ...