An explicit, characteristic-free, equivariant homology equivalence between Koszul complexes

(aka: Divisors over determinantal rings defined by two by two minors)

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Where to find it:

I have posted this talk on my website. Also, a relevant paper and pre-print are available on my website.

The Set up:

Let R be a ring (probably $\mathbb{Z}$ or a field $\boldsymbol{K}$) and E and G be free R-modules of rank e and g, respectively.

We study: the Koszul complex (1):

$$\cdots \to \mathcal{N}(m-1, n-1, p+1) \to \underbrace{\mathrm{Sym}_m E^* \otimes \mathrm{Sym}_n G \otimes \bigwedge^p (E^* \otimes G)}_{\mathcal{N}(m,n,p)}$$

$$\to \mathcal{N}(m+1, n+1, p-1) \to \dots$$

and its homology (which I'll call $\mathrm{H}_{\mathcal{N}}(m,n,p)$). The differential is

$$\partial(V \otimes X \otimes (v_1 \otimes x_1) \wedge \cdots \wedge (v_p \otimes x_p))$$

$$= \sum_{i=1}^{p} (-1)^{i+1} v_i V \otimes x_i X \otimes (v_1 \otimes x_1) \wedge \dots \widehat{(v_i \otimes x_i)} \wedge \cdots \wedge (v_p \otimes x_p),$$

for all $v_1, \dots, v_p$ in E^*, $x_1, \dots, x_p$ in G, $V \in \mathrm{Sym}_m E^*$, and $X \in \mathrm{Sym}_n G$.

The talk has two parts.

- **Part 1.** We describe 3 contexts in which $\mathrm{H}_{\mathcal{N}}(m,n,p)$ arises.
- **Part 2.** We "repair" (1) to make it exact.

A first context in which $\mathrm{H}_{\mathcal{N}}(m,n,p)$ arises: Resolutions of Universal Rings.

This is how I became interested in the subject.

For any triple of parameters e, f, and g, subject to the obvious constraints, Hochster established the existence of a commutative noetherian ring $\mathcal{R}$ and a universal resolution

$$\mathbb{U}: \quad 0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g,$$

such that for any commutative noetherian ring S and any resolution

$$\mathbb{V}: \quad 0 \to S^e \to S^f \to S^g,$$

there exists a unique ring homomorphism $\mathcal{R} \to S$ with $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{R}} S$.

One of the obvious constraints is $g - e + f \geq 0$. We can say something about the border case $f = e + g$.

The universal ring for

$$\mathbb{U}: \quad 0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g,$$

when $f = e + g$, is $\mathcal{R} = \mathfrak{P}/\mathcal{J}$, where

$\mathfrak{P} = \mathbb{Z}[$ the entries of each matrix, and one Buchsbaum-Eisenbud multiplier$]$

and $\mathcal{J}$ sets the entries of the composition equal to zero and makes the multiplier be a multiplier.

Theorem. If $\boldsymbol{K}$ is a field, then every graded summand in the minimal resolution of $\mathcal{R} \otimes_{\mathbb{Z}} \boldsymbol{K}$ by free $\mathfrak{P} \otimes_{\mathbb{Z}} \boldsymbol{K}$-modules involves $\mathrm{H}_{\mathcal{N}}(m,n,p)$ for some m, n, and p.

A more precise version of the result is:

Theorem. If $\boldsymbol{K}$ is a field, then the minimal resolution of $\mathcal{R} \otimes_{\mathbb{Z}} \boldsymbol{K}$ by free $\mathfrak{P} \otimes_{\mathbb{Z}} \boldsymbol{K}$-modules is

$$0 \to \mathbb{X}_{eg+1} \to \cdots \to \mathbb{X}_0 ,$$

with

$$\mathbb{X}_i = \begin{cases} \bigoplus \mathfrak{P} \otimes_{\boldsymbol{K}} \mathrm{H}_{\mathcal{N}}(m,n,p) \otimes_{\boldsymbol{K}} \bigwedge^{m-n+e} \boldsymbol{K}^f [-m-p, -g-n-p] \\ \oplus \\ \mathfrak{P} \otimes_{\boldsymbol{K}} \mathrm{H}_{\mathcal{N}}(0,e,eg-e-i)[-i,-i], \end{cases}$$

where the first sum is taken over all (m,n,p) with $-e \le m-n \le g-1$ and $m+n+p+1 = i$.

A second context where $\mathrm{H}_{\mathcal{N}}(m,n,p)$ arises:

The connection with divisors of determinantal rings.

- Let $\mathcal{P} = \mathrm{Sym}_\bullet(E^* \otimes G)$, a polynomial ring in the eg variables $v_i \otimes x_j$,
- $S = \mathrm{Sym}_\bullet(E^* \oplus G)$, a polynomial ring in $e+g$ variables $v_1, \dots, v_e, x_1, \dots, x_g$, and
- T be the subring $\sum_m \mathrm{Sym}_m E^* \otimes \mathrm{Sym}_m G$ of S. (So, T is the subring $R[\{x_i v_j\}]$ of $S = R[v_1, \dots, v_e, x_1, \dots, x_g]$).

Notice that $v_i \otimes x_j \mapsto v_i x_j$ gives a ring homomorphism $\mathcal{P} \twoheadrightarrow T$ whose kernel is I_2 of the matrix $(v_i \otimes x_j)$. Thus, T is the determinantal ring defined by the 2×2 minors of a generic $e \times g$ matrix.

• Hashimoto proved that if e and g are both at least five, then $\mathrm{Tor}^{\mathcal{P}}_{3,5}(T,\mathbb{Z})$ is not a free $\mathbb{Z}$-module; so, $\dim_{\boldsymbol{K}} \mathrm{Tor}^{\mathcal{P}}_{3,5}(T,\boldsymbol{K})$ **depends on the characteristic of the field $\boldsymbol{K}$.**

• On the other hand, the Koszul complex $\mathcal{P} \otimes_R \bigwedge^{\bullet}(E^* \otimes G)$ is a homogeneous resolution of the $\mathcal{P}$-module R. It follows that

$$\mathrm{Tor}^{\mathcal{P}}_{p,n+p}(T,R) = \mathrm{H}_{\mathcal{N}}(n,n,p).$$

• There is a determinantal interpretation of $\mathrm{H}_{\mathcal{N}}(m,n,p)$, even when $m \neq n$. For each integer s, let M_s be the T-submodule

$$M_s = \sum_{m-n=s} \mathrm{Sym}_m E^* \otimes \mathrm{Sym}_n G$$

of S. View M_s as a graded T-module by giving $\mathrm{Sym}_{n+s} E^* \otimes \mathrm{Sym}_n G$ grade n. The same reasoning we used before shows that

$$\mathrm{Tor}^{\mathcal{P}}_{p,n+p}(M_{m-n},R) = \mathrm{H}_{\mathcal{N}}(m,n,p).$$

We just saw that $\mathrm{Tor}^{\mathcal{P}}_{p,n+p}(M_{m-n},R) = \mathrm{H}_{\mathcal{N}}(m,n,p)$ where T is the determinantal ring $\mathcal{P}/I_2$ and $\mathcal{P}$ is a polynomial ring in eg variables over R.

Take $R = \mathbb{Z}$.

The divisor class group of T is known to be $\mathbb{Z}$ and $s \mapsto [M_s]$ is an isomorphism from $\mathbb{Z} \to \mathrm{C}\ell(T)$. This numbering satisfies

$M_0 = T$,

M_{g-e} is equal to the canonical class of T, and

M_s is a Cohen-Macaulay T-module if and only if $1-e \le s \le g-1$.

Furthermore, if M_s is a Cohen-Macaulay, then the projective dimension of M_s is $\alpha = (e-1)(g-1)$.

A third context in which $\mathrm{H}_{\mathcal{N}}(m,n,p)$ arises: Chessboard complexes.

• Consider all legal rook configurations on an $e \times g$ chessboard – no more than one rook per row, no more than one rook per column.

• Create a simplicial complex $\Delta_{e,g}$. The vertices are the squares of the Chessboard. The simplicies are the legal rook configurations.

• The chain complex that one uses to compute the homology of $\Delta_{e,g}$ is **exactly the same** as one graded strand of complex (1).

• In 1994, Björner, Lovász, Verćia, Živaljević proved that H_2 of $\Delta_{5,5}$ has 3-torsion. This provides an independent proof of Hashimoto's Theorem.

Part 2: We "repair" the complex (1) and make it exact.

Apply $\mathrm{Hom}_R(\_,R)$ to the complex (1):

$$\cdots \to \mathcal{N}(m-1,n-1,p+1) \to \underbrace{\mathrm{Sym}_m E^* \otimes \mathrm{Sym}_n G \otimes \bigwedge^p (E^* \otimes G)}_{\mathcal{N}(m,n,p)}$$

$$\to \mathcal{N}(m+1,n+1,p-1) \to \ldots,$$

to form its dual, complex (2):

$$\cdots \to \mathcal{M}(m+1,n+1,p-1) \to \underbrace{D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*)}_{\mathcal{M}(m,n,p)}$$

$$\to \mathcal{M}(m-1,n-1,p+1) \to \ldots .$$

The differential in complex (2):

$$\cdots \to \mathcal{M}(m+1, n+1, p-1) \to \underbrace{D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*)}_{\mathcal{M}(m,n,p)}$$

$$\to \mathcal{M}(m-1, n-1, p+1) \to \ldots,$$

is given by

$$u^{(m)} \otimes y^{(n)} \otimes Z \mapsto u^{(m-1)} \otimes y^{(n-1)} \otimes (u \otimes y) \wedge Z.$$

• I'll call the homology of (2) at (m, n, p), $\mathrm{H}_{\mathcal{M}}(m, n, p)$.

Class group magic

$$\left(\begin{array}{c} \text{basically}: \quad \operatorname{Ext}^{\alpha}_{\mathcal{P}}(M_s, \mathcal{P}) \cong M_{g-e-s} \\ \text{for} \quad 1-e \le s \le g-1 \quad \text{with} \quad \alpha = (e-1)(g-1) \\ \text{i.e., if} \quad \mathbb{G} \quad \text{resolves} \quad M_s \quad \text{over} \quad \mathcal{P}, \\ \text{then} \quad \mathbb{G}^* = \operatorname{Hom}_{\mathcal{P}}(\mathbb{G}, \mathcal{P}) \quad \text{resolves} \quad M_{g-e-s} \end{array}\right)$$

says that

$$\mathrm{H}_{\mathcal{N}}(m,n,p) \cong \mathrm{H}_{\mathcal{M}}(m',n',p')$$

provided

$$m+m' = g-1, \quad n+n' = e-1, \quad p+p' = \alpha, \quad \text{and}$$

$$1-e \le m-n \le g-1.$$

Fix the above relationship for the rest of the talk.

In the presence of

$$m+m'=g-1, \quad n+n'=e-1, \quad p+p'=\alpha, \quad \text{and}$$

$$1-e \leq m-n \leq g-1,$$

the complexes

$$\mathbb{M}: \qquad \ldots \longrightarrow \mathcal{M}(m,n,p) \longrightarrow \mathcal{M}(m-1,n-1,p+1) \longrightarrow \ldots$$

$$\mathbb{N}: \qquad \ldots \longrightarrow \mathcal{N}(m',n',p') \longrightarrow \mathcal{N}(m'+1,n'+1,p'-1) \longrightarrow \ldots$$

have isomorphic homology. We wonder …

The Main Question

Does there exist a quasi-isomorphism

$$
\begin{array}{lccccccc}
\mathbb{M}: & \ldots & \longrightarrow & \mathcal{M}(m,n,p) & \longrightarrow & \mathcal{M}(m-1,n-1,p+1) & \longrightarrow & \ldots \\
 & & & \big\downarrow & & \big\downarrow & & \\
\mathbb{N}: & \ldots & \longrightarrow & \mathcal{N}(m',n',p') & \longrightarrow & \mathcal{N}(m'+1,n'+1,p'-1) & \longrightarrow & \ldots\,?
\end{array}
$$

Answers

Answer (a). YES!, even when $R = \mathbb{Z}$.

Answer (b). The quasi-isomorphism of Answer (a), depends on the choice of basis. Does there exist a **coordinate free** quasi-isomorphism for the Main Question when $R = \mathbb{Z}$? **NO!**

Answer (c). Does there exist a sequence of coordinate-free quasi-isomorphisms

$$\mathbb{N} \xleftarrow{\ \psi\ } \mathbb{Y} \xrightarrow{\ \varphi\ } \mathbb{M}$$

when $R = \mathbb{Z}$? **YES!**

We create equivariant quasi-isomorphisms:

$$\mathbb{N} \xleftarrow{\ \psi\ } \mathbb{Y} \xrightarrow{\ \varphi\ } \mathbb{M}.$$

Let

$$P = m + p, \quad Q = n + p, \quad P' = m' + p', \quad Q' = n' + p',$$

and $\ell = m - n + e$. Notice that $1 \leq \ell \leq e + g - 1$.

We create $\mathbb{Y}$ to be (a shift of) the total complex of

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathbb{X}_{1,1} & \longrightarrow & \mathbb{X}_{1,0} \\
 & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathbb{X}_{0,1} & \longrightarrow & \mathbb{X}_{0,0}\,.
\end{array}$$

We make $\mathbb{Y}$ so that each row of

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{X}_{1,1} & \longrightarrow & \mathbb{X}_{1,0} & \xrightarrow{\varphi} & \mathcal{M}(m'-n'+1,1,Q'-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{X}_{0,1} & \longrightarrow & \mathbb{X}_{0,0} & \xrightarrow{\varphi} & \mathcal{M}(m'-n',0,Q') & \longrightarrow & 0
\end{array}
$$

is split exact; so,

$$\varphi : \mathbb{Y} \to \mathbb{M}$$

is automatically a quasi-isomorphism.

We define ψ so that the total complex of

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ \cdots & \longrightarrow & \mathbb{X}_{1,1} & \longrightarrow & \mathbb{X}_{1,0} & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{X}_{0,1} & \longrightarrow & \mathbb{X}_{0,0} & & \\ & & \psi \downarrow & & \psi \downarrow & & \\ \cdots & \longrightarrow & \mathcal{N}(\ell-2,e-2,\alpha-Q'+1) & \longrightarrow & \mathcal{N}(\ell-1,e-1,\alpha-Q') & \longrightarrow & \cdots \end{array}$$

is split exact.

The **module** $\mathbb{X}_{r,c}$ is

$$\bigoplus \bigwedge^{\lambda_1} E^* \otimes \cdots \otimes \bigwedge^{\lambda_\ell} E^* \otimes \underbrace{D_{g+r+c} E \otimes D_r G^* \otimes \bigwedge^{\alpha+n'-p-r}(E \otimes G^*)}_{\mathcal{M}(g+r+c,r,\alpha+n'-p-r)},$$

where the sum is taken over all ℓ-tuples $(\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_i \geq 1$ for all i and $\lambda_1 + \cdots + \lambda_\ell = \ell + c$.

The horizontal map $\mathbb{X}_{r,c} \to \mathbb{X}_{r,c-1}$ **is**

$$V_1 \otimes \cdots \otimes V_\ell \otimes u^{(a)} \otimes Y \otimes Z \mapsto$$

$$\sum_i \operatorname{sign}_i \chi(\lambda_i \geq 2) V_1 \otimes \cdots \otimes u(V_i) \otimes \cdots \otimes V_\ell \otimes u^{(a-1)} \otimes Y \otimes Z.$$

The vertical map

$$\begin{array}{ccc} \mathbb{X}_{r,c} & & V\otimes u^{(a)}\otimes y^{(b)}\otimes Z \\ \downarrow & \text{is} & \downarrow \\ \mathbb{X}_{r-1,c} & & V\otimes u^{(a-1)}\otimes y^{(b-1)}\otimes (u\otimes y)\wedge Z. \end{array}$$

The horizontal augmentation

$$X_{r,0} = \underbrace{E^*\otimes\cdots\otimes E^*}_{\ell} \otimes D_aE\otimes D_bG^*\otimes \bigwedge^d (E\otimes G^*) \to \mathcal{M}(P'+r,r,Q'-r)$$

is

$$v_1\otimes\cdots\otimes v_\ell\otimes U\otimes Y\otimes Z \mapsto v_1\cdots v_\ell(U)\otimes Y\otimes Z.$$

I'll tell you the **vertical augmentation**

$$\begin{array}{c} \mathbb{X}_{0,c} \\ \downarrow \psi \\ \mathcal{N}(\ell - 1 - c, e - 1 - c, \alpha - Q' + c) \end{array}$$

by telling you $[\psi(t)](t')$ for each

$$t = V_1 \otimes \cdots \otimes V_\ell \otimes u^{(g+c)} \otimes Y \otimes Z$$

in $\bigwedge^{\lambda_1} E^* \otimes \cdots \otimes \bigwedge^{\lambda_\ell} E^* \otimes \mathcal{M}(g+c, r, \alpha + n' - p) \subset \mathbb{X}_{0,c}$ and

$$t' = U' \otimes Y' \otimes Z' \in \mathcal{M}(\ell - 1 - c, e - 1 - c, \alpha - Q' + c).$$

(a) The value of $[\psi(t)](t')$ is zero unless every $\lambda_i \leq 2$ and $\lambda_\ell = 1$.

(b) Under hypothesis (a), identify $i_1 < \cdots < i_{\ell-c}$ and $j_1 < \cdots < j_c$ with $\lambda_{i_k} = 1$ and $\lambda_{j_k} = 2$ for all k.

(c) The value of $[\psi(t)](t')$ is

$$(V_{i_1} \cdots V_{i_{\ell-c-1}})(U') \\ \cdot \left[Z \wedge Z' \wedge \left(\left(V \wedge V_\ell \right) (\omega_E) \bowtie Y' \right) \wedge \left(u^{(g)} \bowtie \omega_{G^*} \right) \right] (\omega_{E^* \otimes G})$$

for $V = u(V_{j_1}) \wedge \cdots \wedge u(V_{j_c})$.

Recall $t = V_1 \otimes \cdots \otimes V_\ell \otimes u^{(g+c)} \otimes Y \otimes Z$
in $\bigwedge^{\lambda_1} E^* \otimes \cdots \otimes \bigwedge^{\lambda_\ell} E^* \otimes \mathcal{M}(g+c, r, \alpha + n' - p) \subset \mathbb{X}_{0,c}$ and

$$t' = U' \otimes Y' \otimes Z' \in \mathcal{M}(\ell - 1 - c, e - 1 - c, \alpha - Q' + c).$$

The orientation elements ω and the homomorphism
$\bowtie$: $D_a E \otimes \bigwedge^a G^* \to \bigwedge^a (E \otimes G^*)$ are on the next page.

The homomorphism

$$\bowtie: D_aE \otimes \bigwedge^a G^* \to \bigwedge^a (E \otimes G^*)$$

satisfies:

$$u^{(a)} \bowtie (y_1 \wedge \cdots \wedge y_a) = (u \otimes y_1) \wedge \cdots \wedge (u \otimes y_a).$$

Note for the purposes of this talk, I have pretended that all free modules are oriented, that is I have chosen a generator ω_E for $\bigwedge^e E$. This simplifies the exposition, allows us to consider only change of bases which have determinant one, and has no effect on any important idea.

We show an example.

Take $e = g = 2$ and $R = \mathbb{Z}$. I demonstrate that there does not exist a **coordinate-free** quasi-isomorphism

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathcal{M}(0,0,0) & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & \mathcal{N}(1,1,1) & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & 0
\end{array}
$$

I will demonstrate that it is impossible to select a cycle z in $\mathcal{N}(1,1,1)$, such that z is invariant under change of basis and the homology class of z generates all of $\mathrm{H}_{\mathcal{N}}(1,1,1)$.

It is easy to calculate the cycles of $\mathcal{N}(1,1,1)$ which are invariant under under change of basis form the free group generated by the cycle

$$\begin{cases} +v_1 \otimes x_1 \otimes (v_2 \otimes x_2) - v_1 \otimes x_2 \otimes (v_2 \otimes x_1) \\ +v_2 \otimes x_2 \otimes (v_1 \otimes x_1) - v_2 \otimes x_1 \otimes (v_1 \otimes x_2) \end{cases}$$

The given cycle corresponds to 2 in $\mathcal{N}(1,1,1)$. Indeed, each line represents the same generator of $\mathrm{H}_{\mathcal{N}}(1,1,1)$.

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