DIVISORS OVER DETERMINANTAL RINGS DEFINED BY TWO BY TWO MINORS

I have posted this talk on my website. Also, a relevant paper and pre-print are available on my website.

Let R be a ring (probably $\mathbb{Z}$ or a field K) and E and G be free R-modules of rank e and g, respectively. We study the (Koszul) complex

$$\cdots \to \mathcal{N}(m-1,n-1,p+1) \to \underbrace{\operatorname{Sym}_m E^* \otimes \operatorname{Sym}_n G \otimes \textstyle\bigwedge^p (E^* \otimes G)}_{\mathcal{N}(m,n,p)} \to \mathcal{N}(m+1,n+1,p-1) \to \dots, \tag{0.1}$$

its homology (which I'll call $\mathrm{H}_{\mathcal{N}}(m,n,p)$), its dual

$$\cdots \to \mathcal{M}(m+1,n+1,p-1) \to \underbrace{D_m E \otimes D_n G^* \otimes \textstyle\bigwedge^p (E \otimes G^*)}_{\mathcal{M}(m,n,p)} \to \mathcal{M}(m-1,n-1,p+1) \to \dots, \tag{0.2}$$

and the homology of its dual (which I'll call $\mathrm{H}_{\mathcal{M}}(m,n,p)$).

The plan of my talk:

Topic 1 The complex (0.1) can have interesting homology.
Topic 2 The complex (0.1) really is a Koszul complex.
Topic 3 The connection with divisors of determinantal rings.
Topic 3.a The homology of (0.1) depends on characteristic.
Topic 3.b Duality
Topic 4 Does there exist a quasi-isomorphism from (0.2) to (0.1), for the appropriate choice of parameters?
Topic 5 A brand new example.
Topic 6 One step beyond the Cohen-Macaulay range.
Topic 7 Why I first become interested in (0.1).
Topic 8 A pep talk on Divided powers.

Topic 1. The complex (0.1) can have interesting homology.

Consider

$$0 \to \underbrace{\mathcal{N}(0,0,2)}_{\bigwedge^2(E^*\otimes G)} \to \underbrace{\mathcal{N}(1,1,1)}_{E^*\otimes G\otimes(E^*\otimes G)} \to \underbrace{\mathcal{N}(2,2,0)}_{\operatorname{Sym}_2 E^*\otimes \operatorname{Sym}_2 G} \to 0$$

The homology is concentrated in spot $(1,1,1)$. If $v_1, v_2, \dots$ is part of a basis for E^* and $x_1, x_2, \dots$ is part of a basis for G, then

$$v_1 \otimes x_1 \otimes (v_2 \otimes x_2) - v_1 \otimes x_2 \otimes (v_2 \otimes x_1)$$

is a cycle which represents a non-zero element of homology.

In light of duality, $(\mathrm{H}_{\mathcal{N}}(m,n,p) \cong \mathrm{H}_{\mathcal{M}}(m',n',p')$, provided $m+m' = g-1$, $n+n' = e-1$ and $p+p' = (e-1)(g-1)$ and $1-e \le m-n \le g-1)$, the only homology of

$$\dots \to \mathcal{M}(g-1,e-1,\alpha-2) \to \mathcal{M}(g-2,e-2,\alpha-1) \to \mathcal{M}(g-3,e-3,\alpha) \to \dots$$

occurs at $(g-2,e-2,\alpha-1)$ (write α for $(e-1)(g-1)$). Of course, Topic 4 asks if there is a map of complexes

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathcal{M}(g-1,e-1,\alpha-2) & \longrightarrow & \mathcal{M}(g-2,e-2,\alpha-1) & \longrightarrow & \mathcal{M}(g-3,e-3,\alpha) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & \mathcal{N}(1,1,1) & \longrightarrow & \mathcal{N}(2,2,0) & \longrightarrow & 0,
\end{array}$$

so that the mapping cone of the resulting picture is exact.

Topic 2. The complex (0.1) really is a Koszul complex.

Consider the identity map

$$E^* \otimes G \to E^* \otimes G. \tag{0.3}$$

Well,

$$E^* \otimes G = \operatorname{Sym}_1 E^* \otimes \operatorname{Sym}_1 G \subseteq \operatorname{Sym}_2(E^* \oplus G) \subseteq \operatorname{Sym}_\bullet(E^* \oplus G),$$

so (0.3) induces

$$\mathrm{Sym}_\bullet(E^*\oplus G)\otimes E^*\otimes G\to \mathrm{Sym}_\bullet(E^*\oplus G)\otimes E^*\otimes G\xrightarrow{\text{mult}}\mathrm{Sym}_\bullet(E^*\oplus G).$$

We have a map from a free module to a ring, it makes sense to form the usual Koszul complex $\bigwedge^\bullet$(free module) and it makes sense to look at graded strands of this Koszul complex.

If $v_1,\dots,v_e$ is a basis for E^* and $x_1,\dots,x_g$ is a basis for G, then $S=\mathrm{Sym}_\bullet(E^*\oplus G)$ is the polynomial ring $R[v_1,\dots,v_e,x_1,\dots,x_g]$ and (0.1) is a graded strand of the Koszul complex on $\{v_ix_j\}$.

Topic 3. The connection with divisors of determinantal rings.

Let

$$\mathcal{P}=\mathrm{Sym}_\bullet(E^*\otimes G)\quad\text{a polynomial ring in } eg \text{ variables } v_i\otimes x_j,$$

$$S=\mathrm{Sym}_\bullet(E^*\oplus G)\quad\text{a polynomial ring in } e+g \text{ variables } v_1,\dots,v_e,x_1,\dots,x_g,$$

and T be the subring $\sum_m \mathrm{Sym}_m E^*\otimes\mathrm{Sym}_m G$ of S. (So, T is the subring $R[\{x_iv_j\}]$ of $S=R[v_1,\dots,v_e,x_1,\dots,x_g]$).

Notice that $v_i\otimes x_j\mapsto v_ix_j$ gives a ring homomorphism $\mathcal{P}\twoheadrightarrow T$ whose kernel is I_2 of the matrix $(v_i\otimes x_j)$. Thus, T is the determinantal ring defined by the 2×2 minors of a generic $e\times g$ matrix.

The class group of T is $\mathbb{Z}$ with the integer ℓ associated to the divisor $M_\ell=\sum\limits_{m-n=\ell}\mathrm{Sym}_m E^*\otimes\mathrm{Sym}_n G$. (Notice that M_ℓ is a T-submodule of S.)

One can resolve R as a $\mathcal{P}$-module. The resolution is the Koszul complex

$$\mathcal{P}\otimes\bigwedge^\bullet(E^*\otimes G).$$

Apply $M_\ell\otimes_{\mathcal{P}}$ __:

$$\dots\to\sum_{m-n=\ell}\mathrm{Sym}_m E^*\otimes\mathrm{Sym}_n G\otimes\bigwedge^p(E^*\otimes G)\to\dots$$

Take one graded strand:

$$\cdots \to \operatorname{Sym}_m E^* \otimes \operatorname{Sym}_n G \otimes \bigwedge^p (E^* \otimes G) \to \dots$$

We conclude that

$$\mathrm{H}_{\mathcal{N}}(m,n,p) = \operatorname{Tor}^{\mathcal{P}}_{p,?}(M_{m-n}, R).$$

(The grading on M_ℓ is defined so that $? = n+p$.) Thus,

$$\boxed{\mathrm{H}_{\mathcal{N}}(m,n,p) = \operatorname{Tor}^{\mathcal{P}}_{p,n+p}(M_{m-n}, R).}$$

Topic 3.a. The homology of (0.1) depends on characteristic.

If $R = \mathbb{Z}$, then $\mathrm{H}_{\mathcal{N}}(m,n,p)$ is not always free! (If $e, g \geq 5$, then $\mathrm{H}_{\mathcal{N}}(2,2,3)$ has 3-torsion.)

If R is a field, then the dimension of $\mathrm{H}_{\mathcal{N}}(m,n,p)$ depends on the characteristic of R. In particular, $\mathrm{H}_{\mathcal{N}}(2,2,3)$ has larger dimension in characteristic 3, when $e, g \geq 5$, than it has in other characteristics.

Topic 3.b. Duality.

Theorem. *Assume that $1-e \leq m-n \leq g-1$. If $m+m' = g-1$, $n+n' = e-1$, and $p+p' = \alpha$, then*

$$\mathrm{H}_{\mathcal{N}}(m,n,p) \cong \mathrm{H}_{\mathcal{M}}(m',n',p').$$

There are three steps to the proof. In this version of the proof, I assume that $R = K$, but that is not needed ultimately. I also only speak about $\ell = m-n$ with $1-e \leq \ell \leq g-1$.

Step 1. If $1-e \leq \ell \leq g-1$, then M_ℓ is Cohen-Macaulay and

$$M_\ell \cong \operatorname{Ext}^{\alpha}_{\mathcal{P}}(M_{g-e-\ell}, \mathcal{P}).$$

Step 2. If $\mathbb{F}$ resolves M_ℓ, then $\boxed{\mathbb{F}^*[-\alpha, g-eg] \text{ resolves } M_{g-e-\ell}.}$ (I won't justify the internal shift $g-eg$ in public.)

Step 3. Do the calculation.

I do step 3 first. Suppose $\mathcal{P}(-q)^{\beta_{pq}}$ is a summand of the resolution $\mathbb{F}$ of M_ℓ in position p. Then $\mathcal{P}(+q)^{\beta_{pq}}$ is a summand of $\mathbb{F}^*[-\alpha]$ in position $\alpha - p$ and $\mathcal{P}(-(eg-g-q))^{\beta_{pq}}$ is a summand the resolution of $M_{g-e-\ell}$ in position $\alpha - p$. So,

$$\dim \mathrm{H}_{\mathcal{N}}(m,n,p) = \dim \mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, K) = \dim \mathrm{Tor}_{\alpha-p,eg-g-q}(M_{g-e-\ell}, K) = \dim \mathrm{H}_{\mathcal{N}}(m'n',p')$$

where

$$m-n=\ell \quad n+p=q \qquad\qquad p'=\alpha-p \quad m'-n'=g-e-\ell \quad n'+p'=eg-g-q.$$

Add to learn

$$p+p'=\alpha \quad \underbrace{n+n'+\alpha=(e-1)g}_{n+n'=e-1} \quad \underbrace{m+m'-(e-1)=g-e}_{m+m'=g-1}. \quad \square$$

The proof of Step 1. The canonical class is M_{g-e}. The class group arithmetic tells us that

$$M_\ell \cong \mathrm{Hom}_T(M_{g-e-\ell}, M_{g-e}) \cong \mathrm{Hom}_T(M_{g-e-\ell}, \mathrm{Ext}^\alpha_{\mathcal{P}}(T,\mathcal{P}))$$

$$= \mathrm{Ext}^0_{\mathcal{P}}(M_{g-e-\ell}, \mathrm{Ext}^\alpha_{\mathcal{P}}(T,\mathcal{P})) \cong_\bigstar \mathrm{Ext}^0_{\mathcal{P}}(T, \mathrm{Ext}^\alpha_{\mathcal{P}}(M_{g-e-\ell},\mathcal{P}))$$

$$= \mathrm{Hom}_{\mathcal{P}}(T, \mathrm{Ext}^\alpha_{\mathcal{P}}(M_{g-e-\ell},\mathcal{P})) = \mathrm{Ext}^\alpha_{\mathcal{P}}(M_{g-e-\ell},\mathcal{P}).$$

I justify the isomorphism $\bigstar$. Let M and N be perfect $\mathcal{P}$ modules of projective dimension c and let $\mathbb{F}$ and $\mathbb{G}$ be free resolutions of M and N, respectively. The complex

$$\mathrm{Hom}(\mathbb{F}, \mathbb{G}^*) = \mathrm{Hom}(\mathbb{F}\otimes\mathbb{G}, \mathcal{P}) = \mathrm{Hom}(\mathbb{G}, \mathbb{F}^*)$$

shows that

$$\mathrm{Ext}^j_{\mathcal{P}}(M, \mathrm{Ext}^c_{\mathcal{P}}(N,\mathcal{P})) \cong \mathrm{Ext}^j_{\mathcal{P}}(N, \mathrm{Ext}^c_{\mathcal{P}}(M,\mathcal{P}))$$

for all j.

Topic 4. Does there exist a quasi-isomorphism from (0.2) to (0.1), for the appropriate choice of parameters?

Question 0.4. *Suppose* $1-e \leq m-n \leq g-1$. *Choose* m', n', p' *to satisfy* $m+m' = g-1$, $n+n' = e-1$, *and* $p+p' = \alpha$. *Does there exist a quasi-isomorphism*

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{M}(m,n,p) & \longrightarrow & \mathcal{M}(m-1,n-1,p+1) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \mathcal{N}(m',n',p') & \longrightarrow & \mathcal{N}(m'+1,n'+1,p'-1) & \longrightarrow & \dots\ ? \end{array}$$

Answer 1. Yes, even when $R = \mathbb{Z}$.

Answer 2. The quasi-isomorphism of answer 1, depends on the choice of basis. Does there exist a coordinate free quasi-isomorphism for Question 0.4 when $R = \mathbb{Z}$? NO!

Further question 3. Does there exist a coordinate free quasi-isomorphism for Question 0.4 when $R = \mathbb{Q}$? Probably!

Further question 4. Does there exist a sequence of coordinate-free quasi-isomorphisms which connect the two complexes of Question 0.4 when $R = \mathbb{Z}$? I don't know.

Topic 5. We show an example for Answer 2.

Take $e = g = 2$ and $R = \mathbb{Z}$. I demonstrate that there does not exist a coordinate free quasi-isomorphism

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathcal{M}(0,0,0) & \longrightarrow & 0 & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & \mathcal{N}(1,1,1) & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & 0 \end{array}$$

I will demonstrate that it is impossible to select a cycle z in $\mathcal{N}(1,1,1)$, such that z is invariant under change of basis and the homology class of z generates all of

$\mathrm{H}_{\mathcal{N}}(1,1,1)$. It is easy to calculate the cycles of $\mathcal{N}(1,1,1)$ which are invariant under under change of basis. These cycles form the free group generated by the cycle

$$\begin{cases} +v_1 \otimes x_1 \otimes (v_2 \otimes x_2) - v_1 \otimes x_2 \otimes (v_2 \otimes x_1) \\ +v_2 \otimes x_2 \otimes (v_1 \otimes x_1) - v_2 \otimes x_1 \otimes (v_1 \otimes x_2) \end{cases}$$

The given cycle corresponds to 2 in $\mathcal{N}(1,1,1)$.

Topic 6. What happens if one looks at M_ℓ, where ℓ is OUTSIDE of the Cohen-Macaulay range?

In general, I do not know. However, I do know how to glue

$$\mathcal{N}(0,e,p) \to \mathcal{N}(1,e+1,p-1) \to \dots$$

together with

$$\dots \to \mathcal{M}(g+1,1,p'-1) \to \mathcal{M}(g,0,p')$$

to get a coordinate free resolution (for the appropriate choice of p and p'):

$$\dots \longrightarrow \mathcal{M}(g+1,1,p'-1) \longrightarrow \mathcal{M}(g,0,p') \xrightarrow{\text{“}\bowtie\text{”}} \bigwedge^{g+p'}(E \otimes G^*) \xrightarrow{\text{the dual of “}\bowtie\text{”}} \mathcal{N}(0,e,p) \longrightarrow \mathcal{N}(1,e+1,p-1) \longrightarrow \dots$$

For this to make sense, we must have $g+p'+e+p = eg$; i.e., $p+p' = \alpha - 1$.

Topic 7. Why I first become interested in (0.1).

Given e and g, there exists a Universal ring $\mathcal{U}$ and a universal resolution

$$\mathbb{U}\colon 0 \to \mathcal{U}^e \to \mathcal{U}^{e+g} \to \mathcal{U}^g,$$

so that if $\mathcal{V}$ is any ring and

$$\mathbb{V}\colon 0 \to \mathcal{V}^e \to \mathcal{V}^{e+g} \to \mathcal{V}^g$$

is any resolution then there exists a unique ring homomorphism $\mathcal{U} \to \mathcal{V}$ such that $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{U}} \mathcal{V}$. The ring $\mathcal{U}$ is equal to a P/I, where P is a polynomial ring in $(e+g)^2+1$ variables. I set out to find the P-resolution of $\mathcal{U}$. It turns out that every module in the minimal $P \otimes K$-resolution of $\mathcal{U} \otimes K$ is of the form $\mathrm{H}_{\mathcal{M}}(m,n,p) \otimes P \otimes K$ or $\mathrm{H}_{\mathcal{N}}(m,n,p) \otimes P \otimes K$ for some m,n,p.

Topic 8. A pep talk on Divided powers.

The idea is to make $D_\bullet G^*$ be a ring with $D_n G^*$ equal to $\operatorname{Hom}_R(\operatorname{Sym}_n G, R)$. Let $x_1, \dots, x_g$ be a basis for G and $y_1, \dots, y_g$ be the corresponding dual basis for G^*. Now we look for the natural basis for $\operatorname{Hom}_R(\operatorname{Sym}_2 G, R)$ which is dual to $x_1x_2, x_1^2, \dots$. It is reasonable to think of y_1y_2 as a product. It is NOT reasonable to think of the element of $(\operatorname{Sym}_2 G)^*$ which is dual to x_1^2 as a product! If $y_1^{(2)}$ is the name of the dual of x_1^2, then $y_1^2 = 2y_1^{(2)}$.

I have seen people define divided powers by saying that $y^{(n)} = \frac{1}{n!}y^n$. This makes me shudder. One should say something like: figure out all of the rules that $y^{(n)} = \frac{1}{n!}y^n$ satisfies and define divided powers to be the algebraic object which satisfies all of those rules. In particular,

$$(y_1 + y_2)^{(n)} = \sum_{a+b=n} y_1^{(a)} y_2^{(b)} \quad \text{NO Binomial coefficients}$$

and

$$y^{(n)} y^{(m)} = \binom{n+m}{n} y^{(n+m)} \quad \text{Unexpected binomial coefficients}$$

In summary, the multiplication in $D_\bullet G^*$ is unpleasant:

$$\left(y_1^{(a_1)} \dots y_g^{(a_g)}\right)\left(y_1^{(b_1)} \dots y_g^{(b_g)}\right) = \binom{a_1+b_1}{a_1} \dots \binom{a_g+b_g}{a_g} y_1^{(a_1+b_1)} \dots y_g^{(a_g+b_g)},$$

but co-multiplication

$$\Delta : D_n G^* \to \sum_{a+b=n} D_a G^* \otimes D_b G^*$$

is awesome. In particular,

$$\Delta : D_n G^* \to D_1 G^* \otimes D_{n-1} G^*$$

sends

$$y_1^{(a_1)} \dots y_g^{(a_g)} \mapsto \sum_i y_i \otimes y_1^{(a_1)} \dots y_i^{(a_i-1)} \dots y_g^{(a_g)}.$$