REDUCTION NUMBERS
OF PLANAR IDEALS
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• **This is joint work with Claudia Polini and Bernd Ulrich.**

• **We get to work:**

A *reduction* of an ideal I is an ideal $J \subseteq I$ with $JI^i = I^{i+1}$ for all large i.

Reductions were introduced by Northcott and Rees in 1954 to study multiplicities. A reduction can be thought of as a simplification of the ideal I. If $J \subseteq I$, then the following are equivalent:

(a) J is a reduction of I,
(b) $I \subseteq \overline{J}$,
(c) The inclusion of rings $\mathcal{R}(J) = R[Jt] \subseteq R[It] = \mathcal{R}(I)$ is an integral extension.

The *reduction number* of I with respect to the reduction J is

$$r_J(I) = \min\{i \geq 0 \mid JI^i = I^{i+1}\}.$$

A reduction J of I is *minimal* if J does not contain any other reduction of I. The *reduction number* of I is defined by

$$r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

Example. Let

$$B = \frac{k[T_1, \ldots, T_c]}{I_2 \begin{bmatrix} T_1 & \ldots & T_{c-1} & T_c \\ T_2 & \ldots & T_c & P \end{bmatrix}},$$

where P is some linear form in $k[T_1, \ldots, T_c]$. Let $\mathfrak{m} = (T_1, \ldots, T_c)$. We calculate $r(\mathfrak{m})$. Notice that $\mathfrak{m}^2 \subseteq T_1\mathfrak{m}$. Indeed,

$$\begin{gathered} T_2^2 = T_1T_3 \\ T_2T_3 = T_1T_4 \\ \vdots \\ T_2T_c \in (T_1)\mathfrak{m} \\ T_3^2 = T_2T_4 \in (T_1)\mathfrak{m} \\ \vdots \end{gathered}$$

So $(T_1) \subseteq \mathfrak{m}$ is a reduction, $r_{(T_1)}(\mathfrak{m}) = 1$, (T_1) is a minimal reduction, so $r(\mathfrak{m}) \leq 1$; but $r(\mathfrak{m}) \neq 0$, so $r(\mathfrak{m}) = 1$.

• **The example is illuminating in the sense that:** whenever $B = \oplus_{i \geq 0} B_i$ is a standard graded algebra over an infinite field k, then every minimal reduction of $\mathfrak{m}$ is generated by $\dim B$ linear forms in $\mathfrak{m}$. Hence, the notions "minimal reduction of $\mathfrak{m}$" and "linear system of parameters of B" coincide. Furthermore, "most" l.s.o.p.s give rise to $r(\mathfrak{m})$. In other words,

$$(*) \qquad r(\mathfrak{m}) = \operatorname{tsd} B/(\text{general l.s.o.p.}).$$

• Of course, by general I mean something: Let $b_1, \dots, b_N$ generate B_1. Each element of $\mathbb{A}^{N \times D}$ corresponds to a D-tuple of linear forms in B_1. I assert that (*) holds on a dense Zariski open set in $\mathbb{A}^{N \times D}$.

• **But, what if you would like to know $r(I)$ for some non-maximal ideal I in the local or graded ring $(R, \mathfrak{m})$?** Well, it turns out that $r(I) = r(\mathfrak{m}_{\mathcal{F}(I)})$, where

$$\mathcal{F}(I) = \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \cdots = \frac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)}.$$

It remains true that

$$\mathcal{R}(I) = R[It] = R \oplus I \oplus I^2 \oplus I^3 \oplus \dots \ .$$

(I think this was known already by Northcott and Rees. A more modern version is due to Trung. It is in Huneke-Swanson.) So, one can study $r(I)$ for arbitrary ideals I by studying $r(\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal in a standard graded k-algebra; and this question amounts to studying tsd in graded artinian algebras.

• **What ideals do we study?** We study ideals $I \subseteq R = k[x, y]$, with $\operatorname{ht} I = 2$. These ideals are minimally generated by m homogeneous forms of degree d. (The number n satisfies $n + m - 2 = d$.) The other hypothesis on I is one of the following equivalent statments:

(a) the minimal resolution of I is

$$0 \to R(-d-1)^{m-2} \oplus R(-d-n) \to R(-d)^m \to I,$$

or

(b) $I = I_{m-1}(\varphi)$ for some $m \times (m-1)$ matrix φ where every entry in the last column of φ is homogeneous of degree n, every other entry of φ is a linear form and $n + m - 2 = d$, or

(c) there exist non-negative integers $a \geq b$ and homogeneous forms F_1 and F_2, with $a+b=m-2$, $\deg F_1 = d-a$, $\deg F_2 = d-b$, and

$$I = (x,y)^a F_1 + (x,y)^b F_2.$$

The three conditions (a), (b), (c) are equivalent. Take one of them to be the description of I that you like and lets deal with it.

• **Lets find $r(I)$?**

• If $b=0$, then $\mathcal{F}(I)$ is CM; so it is easy to calculate $r(I) = \lceil \frac{n-1}{a} \rceil + 1$.

• If $a|(n-1)$, then one needs an observation that I am not going to make today and the words that I will say are not exactly correct in that case, so I'll just tell you that $r(I) = \lceil \frac{n-1}{a} \rceil + 1$.

• Henceforth, we assume $b > 0$ and $a \not| (n-1)$. In this case, $\mathcal{F}(I) = A/\mathcal{A}$, where A is the RNS ring

$$A = \frac{k[T_1, \dots, T_{a+1}, S_1, \dots, S_{b+1}]}{I_2 \begin{bmatrix} T_1 & \dots & T_a & S_1 & \dots & S_b \\ T_2 & \dots & T_{a+1} & S_2 & \dots & S_{b+1} \end{bmatrix}}, \quad \dim A = 3$$

and $\mathcal{A} \cong K^{(n)}(-1)$ where K is the ideal of A generated by the first row. (The isomorphism is explicitly given by multiplication by an element of the quotient field of A.)

• Furthermore, we have

$$r(I) = r(\mathfrak{m}_{\mathcal{F}(I)}) = r(\mathfrak{m}_{A/K^{(n)}}) + 1 = \operatorname{tsd} \frac{A}{K^{(n)} + (\text{2glf})} + 1.$$

• One observes:

$$\operatorname{MGD} K^{(n)} - 1 \leq_* \operatorname{tsd} \frac{A}{K^{(n)} + (\text{2glf})} \leq_{**} \operatorname{MGD} K^{(n)}.$$

$*$ holds because $\mathfrak{m} \not\subseteq I_2(\ \ ) + (2 \text{ forms})$. One MUST involve $K^{(n)}$.

$**$ holds because we consider a specific specialization. Set $T_{a+1} = S_1$.

• The minimal generator degree of $K^{(n)}$ is $\lceil \frac{n}{a} \rceil$. To calculate the value of the Hilbert function $\operatorname{Hgy}_{K^{(n)}}(\lceil \frac{n}{a} \rceil)$, one simply counts the number of monomials of degree $\lceil \frac{n}{a} \rceil$ that appear in the minimal generating set of $K^{(n)}$. This amounts to counting

$$\begin{aligned} &\{(i,k) \mid i+k = \lceil \tfrac{n}{a} \rceil \quad \text{and} \quad ai < n \leq ai + a + 1 - k\} \\ &\cup \{(i,j,k) \mid i+j+k = \lceil \tfrac{n}{a} \rceil \quad \text{and} \quad ai + bj < n \leq ai + bj + b + 1 - k\}, \end{aligned}$$

where i, j, and k are non-negative integers.

Theorem.
(a) *If* $b = 0$ *or* $a|n-1$, *then* $r(I) = \lceil \frac{n-1}{a} \rceil + 1$.
(b) *Otherwise,* $\lceil \frac{n}{a} \rceil \leq r(I) \leq \lceil \frac{n}{a} \rceil + 1$; *furthermore,*

$$r(I) = \lceil \tfrac{n}{a} \rceil \iff \dim[K^{(n)}]_{\lceil \frac{n}{a} \rceil} \geq m-2.$$

Some ideas from the proof.

($\Rightarrow$) $A/(2 \text{ forms})$ has Hilbert function $1, m-2, m-2, \ldots$. So the image of $[K^{(n)}]_{\mathrm{MGD}}$ must have dimension $m-2$ to make $\left[\frac{A}{K^{(n)}+(2\text{ forms})}\right]_{\mathrm{MGD}}$ vanish. So, $[K^{(n)}]_{\mathrm{MGD}}$ must have dimension at least $m-2$.

($\Leftarrow$) Apply

Socle Lemma of Huneke-Ulrich. *Let k be a field of characteristic zero, R be a polynomial ring over k, and M be graded R-module. Consider the exact sequence*

$$0 \to \mathrm{Ker} \to M(-1) \xrightarrow{\ell} M \to M/(\ell M) \to 0,$$

with ℓ a general linear form and $\mathrm{Ker} \neq 0$. *Then*

$$\mathrm{MGD}(\mathrm{Ker}) > \mathrm{MGD}\left(\mathrm{soc}\left(\frac{M}{\ell M}\right)\right).$$

We use the Socle Lemma to show that

$$\dim\left[\mathrm{Im}\, K^{(n)} \text{ in } \frac{A}{(\text{two forms})}\right]_{\mathrm{MGD}} = [\dim K^{(n)}]_{\mathrm{MGD}}.$$

The Socle Lemma FAILS in characteristic p. Take $p = 2$, $M = \frac{k[x,y,z]}{\mathfrak{m}^{[2]}}$. We have the exact sequence:

$$0 \to \underbrace{\ell M(-1)}_{\mathrm{MGD}=2} \to M(-1) \xrightarrow{\ell} M \to \underbrace{\frac{M}{\ell M}}_{\mathrm{MGD(soc)}=2} \to 0.$$

The proof of the Socle Lemma uses Characteristic zero because:

$\mathfrak{m}^\ell$ is generated by $\{y^\ell \mid y \in R_1\}$ in characteristic zero.

Of course, $\mathfrak{m}^{[2]} \neq (\{y^2 \mid y \in R_1\})$ in characteristic 2.

• We are able to apply the socle lemma in all characteristics because we maneuver ourselves into the ring

$$B = \frac{k'[T_1, \ldots, T_c]}{I_2 \begin{bmatrix} T_1 & \ldots & T_{c-1} & T_c \\ T_2 & \ldots & T_c & \sum \lambda_i T_i \end{bmatrix}}$$

where the λ_i are new variables and k' is the splitting field of

$$\frac{k(\{\lambda_i\})}{\text{(some polynomial with the } \lambda' s \text{ as coefficients)}}.$$

• We prove in B that $\mathfrak{m}^\ell = (\{y^\ell \mid y \in R_1\})$.

• Our proof involves simultaneously diagonalizing $T_1^{-1}T_j \colon B_1 \to B_1$