An explicit, characteristic-free, equivariant homology equivalence between Koszul complexes

(aka: Divisors over determinantal rings defined by two by two minors, II)

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September, 2007

Where to find it:

I have posted this talk on my website. Also, a relevant paper and pre-print are available on my website.

The Set up:

Let R be a ring (probably $\mathbb{Z}$ or a field K) and E and G be free R-modules of rank e and g, respectively.

We study: the (Koszul) complex

$$\cdots \to \mathcal{N}(m-1, n-1, p+1) \to \underbrace{\mathrm{Sym}_m E^* \otimes \mathrm{Sym}_n G \otimes \bigwedge^p (E^* \otimes G)}_{\mathcal{N}(m,n,p)} \qquad (1)$$

$$\to \mathcal{N}(m+1, n+1, p-1) \to \ldots,$$

its homology (which I'll call $\mathrm{H}_{\mathcal{N}}(m, n, p)$), its dual

$$\cdots \to \mathcal{M}(m+1, n+1, p-1) \to \underbrace{D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*)}_{\mathcal{M}(m,n,p)} \qquad (2)$$

$$\to \mathcal{M}(m-1, n-1, p+1) \to \ldots,$$

and the homology of its dual (which I'll call $\mathrm{H}_{\mathcal{M}}(m, n, p)$).

The plan for this talk:

Topic 1. The complex (1) can have interesting homology.

Topic 2. The connection with divisors of determinantal rings.

Topic 2.a. The homology of (1) depends on characteristic.

Topic 2.b. Duality

Topic 3. Does there exist a quasi-isomorphism from (2) to (1), for the appropriate choice of parameters?

Topic 3.a. Yes, but this quasi-isomorphism depends (heavily) on the choice of a basis.

Topic 3.b. Independent of basis? NO. (This is new.)

Topic 3.c. What can be done in an equivariant manner? (This also is new.)

Topic 4. Show an example of 3.b.

Topic 5. Fix 3.b.

Topic 6. Show the answer to 3.c

Topic 7. Ask a question, maybe?

Topic 8. Application 1. We exhibit a generator for $\mathrm{H}_{\mathcal{N}}(g-1, e-1, \alpha-1) \simeq R$.

Topic 9. Application 2. The betti numbers of the divisors of the determinantal rings defined by the 2×2 minors of a generic $e \times g$ matrix exhibit an extra symmetry when $e=3$ or $g=3$.

Topic 10. The proof of 6.

Topic 1. The complex (1) can have interesting homology.

Consider

$$0 \to \underbrace{\mathcal{N}(0,0,2)}_{\bigwedge^2(E^*\otimes G)} \to \underbrace{\mathcal{N}(1,1,1)}_{E^*\otimes G\otimes(E^*\otimes G)} \to \underbrace{\mathcal{N}(2,2,0)}_{\mathrm{Sym}_2 E^*\otimes \mathrm{Sym}_2 G} \to 0$$

The homology is concentrated in spot $(1,1,1)$. If $v_1, v_2, \ldots$ is part of a basis for E^* and $x_1, x_2, \ldots$ is part of a basis for G, then

$$v_1 \otimes x_1 \otimes (v_2 \otimes x_2) - v_1 \otimes x_2 \otimes (v_2 \otimes x_1)$$

is a cycle which represents a non-zero element of homology.

In light of duality:

$$\mathrm{H}_{\mathcal{N}}(m,n,p) \simeq \mathrm{H}_{\mathcal{M}}(m',n',p'),$$

provided

$$m+m'=g-1, \quad n+n'=e-1, \quad p+p'=\underbrace{(e-1)(g-1)}_{\alpha}, \quad \text{and}$$

$$1-e \leq m-n \leq g-1,$$

the only homology of

$$\to \mathcal{M}(g-1,e-1,\alpha-2) \to \mathcal{M}(g-2,e-2,\alpha-1) \to \mathcal{M}(g-3,e-3,\alpha) \to$$

occurs at $(g-2,e-2,\alpha-1)$.

Of course, Topic 3 asks if there is a map of complexes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{M}(g-1,e-1,\alpha-2) & \longrightarrow & \mathcal{M}(g-2,e-2,\alpha-1) & \longrightarrow & \mathcal{M}(g-3,e-3,\alpha) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & \mathcal{N}(1,1,1) & \longrightarrow & \mathcal{N}(2,2,0) & \longrightarrow & 0,
\end{array}$$

so that the mapping cone of the resulting picture is exact.

Topic 2. The connection with divisors of determinantal rings.

- Let $\mathcal{P} = \mathrm{Sym}_\bullet(E^* \otimes G)$, a polynomial ring in the eg variables $v_i \otimes x_j$,
- $S = \mathrm{Sym}_\bullet(E^* \oplus G)$, a polynomial ring in $e+g$ variables $v_1, \ldots, v_e, x_1, \ldots, x_g$, and
- T be the subring $\sum_m \mathrm{Sym}_m E^* \otimes \mathrm{Sym}_m G$ of S. (So, T is the subring $R[\{x_i v_j\}]$ of $S = R[v_1, \ldots, v_e, x_1, \ldots, x_g]$).

Notice that $v_i \otimes x_j \mapsto v_i x_j$ gives a ring homomorphism $\mathcal{P} \twoheadrightarrow T$ whose kernel is I_2 of the matrix $(v_i \otimes x_j)$. Thus, T is the determinantal ring defined by the 2×2 minors of a generic $e \times g$ matrix.

The class group of T is $\mathbb{Z}$ with the integer s associated to the divisor $M_s = \sum\limits_{m-n=s} \mathrm{Sym}_m E^* \otimes \mathrm{Sym}_n G$. (Notice that M_s is a T-submodule of S.)

One can resolve R as a $\mathcal{P}$-module. The resolution is the Koszul complex

$$\mathcal{P} \otimes \bigwedge^{\bullet}(E^* \otimes G).$$

Apply $M_s \otimes_{\mathcal{P}} \_$:

$$\cdots \to \sum_{m-n=s} \mathrm{Sym}_m E^* \otimes \mathrm{Sym}_n G \otimes \bigwedge^p (E^* \otimes G) \to \ldots$$

Take one graded strand:

$$\cdots \to \mathrm{Sym}_m E^* \otimes \mathrm{Sym}_n G \otimes \bigwedge^p (E^* \otimes G) \to \ldots$$

We conclude that

$$\mathrm{H}_{\mathcal{N}}(m,n,p) = \mathrm{Tor}^{\mathcal{P}}_{p,?}(M_{m-n}, R).$$

(The grading on M_s is defined so that $? = n+p$.) Thus,

$$\boxed{\mathrm{H}_{\mathcal{N}}(m,n,p) = \mathrm{Tor}^{\mathcal{P}}_{p,n+p}(M_{m-n}, R).}$$

Topic 2.a. The homology of (1) depends on characteristic.

- If $R = \mathbb{Z}$, then $\mathrm{H}_{\mathcal{N}}(m,n,p)$ is not always free! (If $e,g \geq 5$, then $\mathrm{H}_{\mathcal{N}}(2,2,3)$ has 3-torsion.) **[Hashimoto 1990]**
- If R is a field, then the dimension of $\mathrm{H}_{\mathcal{N}}(m,n,p)$ depends on the characteristic of R. In particular, $\mathrm{H}_{\mathcal{N}}(2,2,3)$ has larger dimension in characteristic 3, when $e,g \geq 5$, than it has in other characteristics.

Topic 2.b. Duality.

Theorem. Assume $1-e \leq m-n \leq g-1$. If

- $m+m' = g-1$,
- $n+n' = e-1$, and
- $p+p' = \alpha$,

then

$$\mathrm{H}_{\mathcal{N}}(m,n,p) \simeq \mathrm{H}_{\mathcal{M}}(m',n',p').$$

Proof. The key step in the proof is: If $1-e \leq s \leq g-1$, then M_s is Cohen-Macaulay and

$$M_s \simeq \mathrm{Ext}^{\alpha}_{\mathcal{P}}(M_{g-e-s}, \mathcal{P}).$$

Once one has this, then the rest is booking keeping (which I will hide).

Show $M_s \simeq \mathrm{Ext}^{\alpha}_{\mathcal{P}}(M_{g-e-s}, \mathcal{P})$.

The canonical class is M_{g-e}. The class group arithmetic tells us that

$$M_s \simeq \mathrm{Hom}_T(M_{g-e-s}, M_{g-e}) \simeq \mathrm{Hom}_T(M_{g-e-s}, \mathrm{Ext}^{\alpha}_{\mathcal{P}}(T, \mathcal{P}))$$

$$= \mathrm{Ext}^0_{\mathcal{P}}(M_{g-e-s}, \mathrm{Ext}^{\alpha}_{\mathcal{P}}(T, \mathcal{P})) \simeq_{\bigstar} \mathrm{Ext}^0_{\mathcal{P}}(T, \mathrm{Ext}^{\alpha}_{\mathcal{P}}(M_{g-e-s}, \mathcal{P}))$$

$$= \mathrm{Hom}_{\mathcal{P}}(T, \mathrm{Ext}^{\alpha}_{\mathcal{P}}(M_{g-e-s}, \mathcal{P})) = \mathrm{Ext}^{\alpha}_{\mathcal{P}}(M_{g-e-s}, \mathcal{P}).$$

I justify the isomorphism $\bigstar$. Let M and N be perfect $\mathcal{P}$ modules of projective dimension c and let $\mathbb{F}$ and $\mathbb{G}$ be free resolutions of M and N, respectively. The complex

$$\mathrm{Hom}(\mathbb{F}, \mathbb{G}^*) = \mathrm{Hom}(\mathbb{F} \otimes \mathbb{G}, \mathcal{P}) = \mathrm{Hom}(\mathbb{G}, \mathbb{F}^*)$$

shows that

$$\mathrm{Ext}^j_{\mathcal{P}}(M, \mathrm{Ext}^c_{\mathcal{P}}(N, \mathcal{P})) \simeq \mathrm{Ext}^j_{\mathcal{P}}(N, \mathrm{Ext}^c_{\mathcal{P}}(M, \mathcal{P}))$$

for all j. □

Topic 3. Does there exist a quasi-isomorphism from (2) to (1), for the appropriate choice of parameters?

Main Question. Suppose $1-e \leq m-n \leq g-1$. Choose m', n', p' to satisfy $m+m' = g-1$, $n+n' = e-1$, and $p+p' = \alpha$. Does there exist a quasi-isomorphism

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \mathcal{M}(m,n,p) & \longrightarrow & \mathcal{M}(m-1,n-1,p+1) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & \mathcal{N}(m',n',p') & \longrightarrow & \mathcal{N}(m'+1,n'+1,p'-1) & \longrightarrow & \ldots ? \end{array}$$

Answer (a). Yes, even when $R = \mathbb{Z}$.

Answer (b). The quasi-isomorphism of Answer (a), depends on the choice of basis. Does there exist a coordinate free quasi-isomorphism for the Main Question when $R = \mathbb{Z}$? NO! (new)

Answer (c). Does there exist a sequence of coordinate-free quasi-isomorphisms

$$\mathbb{N} \xleftarrow{\ \psi\ } \mathbb{Y} \xrightarrow{\ \varphi\ } \mathbb{M}$$

when $R = \mathbb{Z}$? YES! (new)

Topic 4. We show an example for Answer 3.b.

Take $e = g = 2$ and $R = \mathbb{Z}$. I demonstrate that there does not exist a **coordinate-free** quasi-isomorphism

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathcal{M}(0,0,0) & \longrightarrow & 0 & & \\ & & & & \Big\downarrow & & & & \\ 0 & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & \mathcal{N}(1,1,1) & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & 0 \end{array}$$

I will demonstrate that it is impossible to select a cycle z in $\mathcal{N}(1,1,1)$, such that z is invariant under change of basis and the homology class of z generates all of $\mathrm{H}_{\mathcal{N}}(1,1,1)$.

It is easy to calculate the cycles of $\mathcal{N}(1,1,1)$ which are invariant under under change of basis form the free group generated by the cycle

$$\begin{cases} +v_1 \otimes x_1 \otimes (v_2 \otimes x_2) - v_1 \otimes x_2 \otimes (v_2 \otimes x_1) \\ +v_2 \otimes x_2 \otimes (v_1 \otimes x_1) - v_2 \otimes x_1 \otimes (v_1 \otimes x_2) \end{cases}$$

The given cycle corresponds to 2 in $\mathcal{N}(1,1,1)$. The green summand and the red summand each represent generators of $\mathrm{H}_{\mathcal{N}}(1,1,1)$.

(Note for the purposes of this talk, I have pretended that all free modules are oriented, that is I have chosen a generator ω_E for $\bigwedge^e E$. This simplifies the exposition, allows us to consider only change of bases which have determinant one, and has no effect on any important idea.)

Topic 5. Fix 3.b.

It is easy to "fix" 3.b. We merely look for a complex $\mathbb{Y}$ with $\mathrm{H}_0(\mathbb{Y}) = \mathcal{M}(0,0,0)$ for which the mapping cone of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Y}_1 & \longrightarrow & \mathbb{Y}_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{N}(0,0,2) & \longrightarrow & \mathcal{N}(1,1,1) & \longrightarrow & \mathcal{N}(2,2,0) & \longrightarrow & 0
\end{array}$$

is exact.

One such $\mathbb{Y}$ is

$$0 \to \bigwedge^2 E^* \otimes D_2 E \to E^* \otimes E$$

Topic 6. We create equivariant quasi-isomorphisms:

$$\mathbb{N} \xleftarrow{\psi} \mathbb{Y} \xrightarrow{\varphi} \mathbb{M}.$$

- Fix P, Q with $1 \le \underbrace{P - Q + e}_{\ell} \le e + g - 1$.

- Define $P + P' = e(g-1)$ and $Q + Q' = (e-1)g$.

- Let $\mathbb{N} = \mathbb{N}(P,Q)$: $\quad \cdots \to \mathcal{N}(P-1, Q-1, 1) \to \underbrace{\mathcal{N}(P,Q,0)}_{\text{position } 0} \to 0$, and

- $\mathbb{M} = \mathbb{M}(P', Q')[-\alpha]$: $\quad 0 \to \underbrace{\mathcal{M}(P', Q', 0)}_{\text{position } \alpha} \to \mathcal{M}(P'-1, Q'-1, 1) \to \ldots$

We create $\mathbb{Y}$ to be (a shift of) the total complex of

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathbb{X}_{1,1} & \longrightarrow & \mathbb{X}_{1,0} \\ & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathbb{X}_{0,1} & \longrightarrow & \mathbb{X}_{0,0}\,. \end{array}$$

We make $\mathbb{Y}$ so that each row of

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{X}_{1,1} & \longrightarrow & \mathbb{X}_{1,0} & \xrightarrow{\varphi} & \mathcal{M}(P'-Q'+1,1,Q'-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{X}_{0,1} & \longrightarrow & \mathbb{X}_{0,0} & \xrightarrow{\varphi} & \mathcal{M}(P'-Q',0,Q') & \longrightarrow & 0
\end{array}
$$

is split exact; so,

$$\varphi : \mathbb{Y} \to \mathbb{M}$$

is automatically a quasi-isomorphism.

We define ψ so that the total complex of

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \mathbb{X}_{1,1} & \longrightarrow & \mathbb{X}_{1,0} & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{X}_{0,1} & \longrightarrow & \mathbb{X}_{0,0} & & \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \\
\cdots & \longrightarrow & \mathcal{N}(g-2+Q'-P', e-2, \alpha-Q'+1) & \longrightarrow & \mathcal{N}(g-1+Q'-P', e-1, \alpha-Q') & \longrightarrow & \cdots
\end{array}
$$

is split exact.

The module $\mathbb{X}_{r,c}$ is

$$\bigoplus \bigwedge^{\lambda_1} E^* \otimes \cdots \otimes \bigwedge^{\lambda_\ell} E^* \otimes \underbrace{D_{g+r+c} E \otimes D_r G^* \otimes \bigwedge^{\alpha+e-1-Q-r} (E \otimes G^*)}_{\mathcal{M}(g+r+c, r, \alpha+e-1-Q-r)},$$

where the sum is taken over all ℓ-tuples $(\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_i \geq 1$ for all i and $\lambda_1 + \cdots + \lambda_\ell = \ell + c$.

The horizontal map $\mathbb{X}_{r,c} \to \mathbb{X}_{r,c-1}$ is

$$V_1 \otimes \cdots \otimes V_\ell \otimes u^{(a)} \otimes Y \otimes Z \mapsto$$

$$\sum_i \text{sign}_i \chi(\lambda_i \geq 2) V_1 \otimes \cdots \otimes u(V_i) \otimes \cdots \otimes V_\ell \otimes u^{(a-1)} \otimes Y \otimes Z.$$

The vertical map

$$\begin{array}{ccc} \mathbb{X}_{r,c} & & V \otimes u^{(a)} \otimes y^{(b)} \otimes Z \\ \downarrow & \text{is} & \downarrow \\ \mathbb{X}_{r-1,c} & & V \otimes u^{(a-1)} \otimes y^{(b-1)} \otimes (u \otimes y) \wedge Z. \end{array}$$

The horizontal augmentation

$$X_{r,0} = \underbrace{E^* \otimes \cdots \otimes E^*}_{\ell} \otimes D_a E \otimes D_b G^* \otimes \bigwedge^d (E \otimes G^*) \to \mathcal{M}(P' + r, r, Q' - r)$$

is

$$v_1 \otimes \cdots \otimes v_\ell \otimes U \otimes Y \otimes Z \mapsto v_1 \cdots v_\ell(U) \otimes Y \otimes Z.$$

I'll tell you the vertical augmentation

$$\begin{array}{c} \mathbb{X}_{0,c} \\ \Big\downarrow \psi \\ \mathcal{N}(g-1+Q'-P'-c, e-1-c, \alpha-Q'+c) \end{array}$$

by telling you $[\psi(t)](t')$ for each

$$t = V_1 \otimes \cdots \otimes V_\ell \otimes u^{(g+c)} \otimes Y \otimes Z$$

in $\bigwedge^{\lambda_1} E^* \otimes \cdots \otimes \bigwedge^{\lambda_\ell} E^* \otimes \mathcal{M}(g+c, r, \alpha+e-1-Q) \subset \mathbb{X}_{0,c}$ and

$$t' = U' \otimes Y' \otimes Z' \in \mathcal{M}(\underbrace{g-1+Q'-P'-c}_{\ell-1-c}, e-1-c, \alpha-Q'+c).$$

(a) The value of $[\psi(t)](t')$ is zero unless every $\lambda_i \le 2$ and $\lambda_\ell = 1$.

(b) Under hypothesis (a), identify $i_1 < \cdots < i_{\ell-c}$ and $j_1 < \cdots < j_c$ with $\lambda_{i_k} = 1$ and $\lambda_{j_k} = 2$ for all k.

(c) The value of $[\psi(t)](t')$ is

$$(V_{i_1} \cdots V_{i_{\ell-c-1}})(U') \cdot \Big[Z \wedge Z' \wedge \Big(\Big(V \wedge V_\ell \Big) (\omega_E) \bowtie Y' \Big) \wedge \Big(u^{(g)} \bowtie \omega_{G^*} \Big) \Big] (\omega_{E^* \otimes G})$$

for $V = u(V_{j_1}) \wedge \cdots \wedge u(V_{j_c})$.

Recall $t = V_1 \otimes \cdots \otimes V_\ell \otimes u^{(g+c)} \otimes Y \otimes Z$ in $\bigwedge^{\lambda_1} E^* \otimes \cdots \otimes \bigwedge^{\lambda_\ell} E^* \otimes \mathcal{M}(g+c, r, \alpha + e - 1 - Q) \subset \mathbb{X}_{0,c}$ and

$$t' = U' \otimes Y' \otimes Z' \in \mathcal{M}(\ell - 1 - c, e - 1 - c, \alpha - Q' + c).$$

The homomorphism $\bowtie$: $D_a E \otimes \bigwedge^a G^* \to \bigwedge^a (E \otimes G^*)$ is on the next page.

The homomorphism

$$\bowtie: D_a E \otimes \bigwedge^a G^* \to \bigwedge^a (E \otimes G^*)$$

satisfies:

$$u^{(a)} \bowtie (y_1 \wedge \cdots \wedge y_a) = (u \otimes y_1) \wedge \cdots \wedge (u \otimes y_a).$$

Topic 7. Ask a question, maybe?

Let R be a ring – at this point, “ring” should be interpreted in a liberal enough manner to include the possibility $R = \mathbb{Z}[\mathrm{GL}(E) \times \mathrm{GL}(G)]$.

Definition. The complexes $\mathbb{F}$ and $\mathbb{F}'$ are *homologically equivalent* if there exists a sequence of quasi-isomorphisms

$$\mathbb{F} \leftarrow Q_1, \quad Q_1 \rightarrow Q_2, \quad Q_2 \leftarrow Q_3, \quad \dots, \quad Q_t \rightarrow \mathbb{F}'. \tag{3}$$

Question. Let $\mathbb{F}$ and $\mathbb{F}'$ be complexes of free R-modules. Find necessary and sufficient conditions so that $\mathbb{F}$ and $\mathbb{F}'$ are homologically equivalent.

One necessary condition is that

$$\mathrm{H}_i(\mathbb{F}) \simeq \mathrm{H}_i(\mathbb{F}') \tag{4}$$

be isomorphic, for all i.

(a) Is condition (4) sufficient? Find a counterexample.

(b) Is there some significance to the least number t in (3)?

Topic 8. Application 1. We exhibit a generator for $\mathrm{H}_{\mathcal{N}}(g-1, e-1, \alpha-1) \simeq R$.

When $P = e(g-1)$ and $Q = (e-1)g$, then we can use our homology equivalence:

$$\begin{array}{ccc} \underbrace{E^* \otimes \cdots \otimes E^*}_{g} \otimes D_g E \otimes D_0 G^* \otimes \bigwedge^0 (E \otimes G^*) = \mathbb{X}_{0,0} & \xrightarrow{\varphi} & \mathcal{M}(,0,0,0) \\ \Big\downarrow \psi & & \\ \mathcal{N}(g-1, e-1, \alpha) & & \end{array}$$

to pick out a generator for $\mathrm{H}_{\mathcal{N}}(g-1, e-1, \alpha-1) \simeq R$.

There exists a homomorphism

$$\Phi : D_{e-1}G^* \otimes D_{g-1}E \to \bigwedge^{e+g-1}(E \otimes G^*)$$

which satisfies:

(a) $\Phi(x(Y') \otimes U) = (\omega_E \bowtie Y') \wedge (U \bowtie x(\omega_{G^*}))$ for $Y' \in D_e G^*$ and $x \in G$, and

(b) $\Phi(Y \otimes v(U')) = (-1)^{e+1}(v(\omega_E) \bowtie Y) \wedge (U' \bowtie \omega_{G^*})$ for $U' \in D_g E$ and $v \in E^*$.

Exhibit a generator for $\mathrm{H}_{\mathcal{N}}(g-1, e-1, \alpha-1) \simeq R$.

- Take any $s \in \mathrm{Sym}_{g-1} E^* \otimes D_{g-1}E$, which is sent to 1 under the evaluation map.

- Define $\zeta_s \in \mathcal{N}(g-1, e-1, \alpha)$ by

$$\zeta_s(U' \otimes Y' \otimes Z') = [Z' \wedge \Phi(Y' \otimes (U' \otimes 1)(s))](\omega_{E^* \otimes G})$$

  for all $U' \otimes Y' \otimes Z' \in \mathcal{M}(g-1, e-1, \alpha)$.

- The cycle $\zeta_s \in \mathcal{N}(g-1, e-1, \alpha)$ depends on the choice of s.

- The homology class of $\zeta_s \in \mathrm{H}_{\mathcal{N}}(g-1, e-1, \alpha)$ is independent of s and generates $\mathrm{H}_{\mathcal{N}}(g-1, e-1, \alpha)$.

Topic 9. Application 2. The betti numbers of the divisors of the determinantal rings defined by the 2×2 minors of a generic $e \times g$ matrix exhibit an extra symmetry when $e = 3$ or $g = 3$.

Return to the determinantal ring set up. Let K be a field, $\mathcal{P}$ be a polynomial ring over K in eg variables, T be the determinantal ring defined by the 2×2 minors of a generic $e \times g$ matrix, and $\{M_s\}$ be the previously identified set of representatives of the class group of T.

One can translate

$$\mathrm{H}_{\mathcal{N}}(m,n,p) \simeq \mathrm{H}_{M}(m',n',p')$$
$$\text{for } m+m' = g-1,\ n+n' = e-1 \text{ and } p+p' = \alpha$$

into:

$$\beta_{p,q}(M_s) = \beta_{p',q'}(M_{s'})\text{for}$$
$$1-e \leq s \leq g-1,\ s+s' = g-e,\ p+p' = \alpha,\ q+q' = (e-1)g. \tag{5}$$

One also gets

Corollary.

(a) Assume that $e = 3$. If s and q are integers with $-2 \leq s \leq q-1$, then

$$\beta_{p,q}(M_s) = \beta_{p',q}(M_{s'}) \tag{6}$$

for $s + s' = q - 3$ and $p + p' = 2q - 2$.

(b) Assume that $g = 3$. If s and q are integers with $s \leq 2$ and $1 - 2s \leq q$, then

$$\beta_{p,q}(M_s) = \beta_{p',q'}(M_{s'}), \tag{6}$$

provided

$$s + s' = 3 - s - q, \quad q + q' = 3(s + q - 1), \quad \text{and} \quad p + p' = 2(s + q - 1).$$

Example. Take $e = g = 3$. The graded betti numbers of the $\mathcal{P}$-modules M_s, for $-5 \leq s \leq 5$ are

$^{\ddagger}\beta_{0,5}(M_{-5})=21$	$^{\ddagger}\beta_{0,4}(M_{-4})=15$	$^{\ddagger}\beta_{0,3}(M_{-3})=10$	$\beta_{0,2}(M_{-2})=6$
$^{\ddagger}\beta_{1,6}(M_{-5})=105$	$^{\ddagger}\beta_{1,5}(M_{-4})=72$	$^{\ddagger}\beta_{1,4}(M_{-3})=45$	$\beta_{1,3}(M_{-2})=24$
$^{\ddagger}\beta_{2,7}(M_{-5})=216$	$^{\ddagger}\beta_{2,6}(M_{-4})=141$	$^{\ddagger}\beta_{2,5}(M_{-3})=81$	$\beta_{2,4}(M_{-2})=36$
$^{\ddagger}\beta_{3,8}(M_{-5})=234$	$^{\ddagger}\beta_{3,7}(M_{-4})=144$	$^{\ddagger}\beta_{3,6}(M_{-3})=74$	$\beta_{3,5}(M_{-2})=24$
$^{\ddagger}\beta_{4,9}(M_{-5})=141$	$^{\ddagger}\beta_{4,8}(M_{-4})=81$	$\beta_{4,7}(M_{-3})=36$	$\beta_{4,6}(M_{-2})=6$
$^{\ddagger}\beta_{5,10}(M_{-5})=45$	$\beta_{5,9}(M_{-4})=24$	$\beta_{5,8}(M_{-3})=9$	
$\beta_{6,11}(M_{-5})=6$	$\beta_{6,10}(M_{-4})=3$	$\beta_{6,9}(M_{-3})=1$	

$\beta_{0,1}(M_{-1})=3$	$\beta_{0,0}(M_0)=1$	$\beta_{0,0}(M_1)=3$	$\beta_{0,0}(M_2)=6$
$\beta_{1,2}(M_{-1})=9$	$\beta_{1,2}(M_0)=9$	$\beta_{1,1}(M_1)=9$	$\beta_{1,1}(M_2)=24$
$\beta_{2,3}(M_{-1})=6$	$\beta_{2,3}(M_0)=16$	$\beta_{2,2}(M_1)=6$	$\beta_{2,2}(M_2)=36$
$\beta_{2,4}(M_{-1})=6$	$\beta_{3,4}(M_0)=9$	$\beta_{2,3}(M_1)=6$	$\beta_{3,3}(M_2)=24$
$\beta_{3,5}(M_{-1})=9$	$\beta_{4,6}(M_0)=1$	$\beta_{3,4}(M_1)=9$	$\beta_{4,4}(M_2)=6$
$\beta_{4,6}(M_{-1})=3$		$\beta_{4,5}(M_1)=3$	

$^{\ddagger}\beta_{0,0}(M_3)=10$	$^{\ddagger}\beta_{0,0}(M_4)=15$	$^{\ddagger}\beta_{0,0}(M_5)=21$
$^{\ddagger}\beta_{1,1}(M_3)=45$	$^{\ddagger}\beta_{1,1}(M_4)=72$	$^{\ddagger}\beta_{1,1}(M_5)=105$
$^{\ddagger}\beta_{2,2}(M_3)=81$	$^{\ddagger}\beta_{2,2}(M_4)=141$	$^{\ddagger}\beta_{2,2}(M_5)=216$
$^{\ddagger}\beta_{3,3}(M_3)=74$	$^{\ddagger}\beta_{3,3}(M_4)=144$	$^{\ddagger}\beta_{3,3}(M_5)=234$
$\beta_{4,4}(M_3)=36$	$^{\ddagger}\beta_{4,4}(M_4)=81$	$^{\ddagger}\beta_{4,4}(M_5)=141$
$\beta_{5,5}(M_3)=9$	$\beta_{5,5}(M_4)=24$	$^{\ddagger}\beta_{5,5}(M_5)=45$
$\beta_{6,6}(M_3)=1$	$\beta_{6,6}(M_4)=3$	$\beta_{6,6}(M_5)=6.$

These numbers were calculated by the computer program Macaulay. There are four symmetries running through this set of numbers. In addition to the three mentioned above, the R-modules E^* and G are isomorphic because $e = g$; hence, the R-module automorphism of $\mathcal{P}$, which sends the matrix of indeterminates Z to Z^{T}, induces the relation

$$\beta_{p,q}(M_s) = \beta_{p,q+s}(M_{-s}), \tag{7}$$

for all integers p, q, and s. We have marked (with ‡) the betti numbers which satisfy only (7). Each of the other betti numbers also satisfy at least one of the other symmetries. The module M_s is Cohen-Macaulay for $-2 \leq s \leq 2$, and the symmetries of (5) apply only in this range. It is interesting to notice that the symmetries of (6) apply to some of the betti numbers of modules M_s which are not Cohen-Macaulay.

Topic 10. The proof that $\psi : \mathbb{Y} \to \mathbb{N}$ is a quasi-isomorphism.

Let $\mathbb{C}(P,Q)$ be the mapping cone of ψ. We want to prove that all relevant $\mathbb{C}(P,Q)$ are split exact. We proceed by induction on ℓ and g.

• The case $g = 0$ makes sense, is easy, and is exact.

• In today's talk $1 \le \ell \le e+g-1$. (Recall that $\ell = s+e$ and M_s is Cohen-Macaulay and the critical duality

$$M_s \simeq \operatorname{Ext}^{\alpha}_{\mathcal{P}}(M_{g-e-s}, \mathcal{P})$$

holds precisely when $1-e \le s \le g-1$.) Nonetheless, I studied the case "$\ell = 0$" (the module M_{-e} is just outside the Cohen-Macaulay range) in some previous work and one can use "Φ" to connect the cases "$\ell = 0$" and $\ell = 1$ and prove that $\mathbb{C}(P,Q)$ is exact whenever $\ell = 1$. (Recall that $\ell = P - Q + e$.)

Now one decomposes G as $\bar{G} \oplus Rx_g$. Use

$$x_g : \mathcal{N}(a,b,d) \to \mathcal{N}(a,b+1,d) \quad \text{and}$$
$$x_g : \mathcal{M}(a,b,d) \to \mathcal{M}(a,b-1,d) \tag{8}$$

to create a map

$$\Theta : \mathbb{C}(P,Q) \to \mathbb{C}(P,Q+1)[-1].$$

The complex $\mathbb{C}(P,Q+1)[-1]$ is split exact by induction on ℓ; so it suffices to show that the mapping cone, $\mathbb{A}$, of Θ is split exact. Use (8) to split as much from $\mathbb{A}$ as possible. What remains is a direct sum of complexes

$$\bigoplus_{w=0}^{e} \mathbb{C}_{E,\bar{G}}(P-w, Q-w+1)$$

formed using the free module $\bar{G}$ of rank $g-1$. □