A Case Study in Bigraded Commutative Algebra

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Outline

- Graded vs Bigraded
- The Syzygies
- Two Hilbert Functions
- Two Free Resolutions

Joint work with
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Graded Case

If $f_0, f_1, f_2 \in R = k[x_0, x_1, x_2]$ are homogeneous of positive degree, then TFAE:

1. The f_i don't vanish simultaneously on $\mathbb{P}^2$.
2. $\mathrm{Res}(f_0, f_1, f_2) \neq 0$.
3. The f_i form a regular sequence in R.
4. The Koszul complex of $I = \langle f_0, f_1, f_2 \rangle$ is exact:

$$0 \to R \xrightarrow{\begin{bmatrix} f_2 \\ -f_1 \\ f_0 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} f_1 & f_2 & 0 \\ -f_0 & 0 & f_2 \\ 0 & -f_0 & -f_1 \end{bmatrix}} R^3 \xrightarrow{[f_0 \quad f_1 \quad f_2]} I \to 0.$$

Bigraded Case

If $f_0, f_1, f_2 \in R = k[x, y, z, w]$, x, y bidegree $(1, 0)$, z, w bidegree $(0, 1)$, are bihomogeneous of positive bidegree, then TFAE:

1. The f_i don't vanish simultaneously on $\mathbb{P}^1 \times \mathbb{P}^1$.
2. $\mathrm{Res}(f_0, f_1, f_2) \neq 0$.

When this happens,

- The f_i do not form a regular sequence in R.
- The Koszul complex of the f_i is never exact.

We will study $I = \langle f_0, f_1, f_2 \rangle$ for $\deg(f_i) = (2, 1)$.

Syzygies

Let $R_{m,n}, I_{m,n}$ denote graded pieces in bidegree (m, n). With our assumption on the f_i, we get

$$0 \to R_{m-6,n-3} \to R^3_{m-4,n-2} \to R^3_{m-2,n-1} \to I_{m,n} \to 0$$

Exactness fails at the term marked in red.

Let $\mathrm{Syz}(f)$ be the syzygy module of the f_i. Thus

$$0 \to \mathrm{Syz}(f)_{m,n} \to R^3_{m-2,n-1} \xrightarrow{[f_0 \quad f_1 \quad f_2]} I_{m,n} \to 0$$

is exact. Most $\mathrm{Syz}(f)_{m,n}$ are easy to compute.

Picture of $\mathrm{Syz}(f)_{m,n}$

n	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
5	0	0	?	□	□	●	●	●	$\cdots$
4	0	0	?	□	□	●	●	●	$\cdots$
3	0	0	?	□	□	●	●	●	$\cdots$
2	0	0	0	0	●	●	●	●	$\cdots$
1	0	0	0	0	0	0	□	□	$\cdots$
0	0	0	0	0	0	0	0	0	$\cdots$
	0	1	2	3	4	5	6	7	m

□ is non-Koszul

● is Koszul

? is unknown

generator location in red

Examples & Luck

Example 1. $f_0, f_1, f_2 \in R_{2,1}$ chosen randomly.
Here $\mathrm{Syz}(f)$ has no generator of bidegree $(2, 3)$.
In fact, all question marks $?$ are zero!

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Example 2. $f_0 = x^2 z$, $f_1 = y^2 w$, $f_2 = x^2 w + y^2 z$ have no common zeros on $\mathbb{P}^1 \times \mathbb{P}^1$ and satisfy

$$w^2 f_0 + z^2 f_1 - zw\, f_2 = 0.$$

Thus $(w^2, z^2, -zw)$ lies in $\mathrm{Syz}(p)_{2,3} \subseteq R_{0,2}^3$.

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This was the first example we computed!

There are at least two free resolutions!

Some Geometry

The polynomials $f_0, f_1, f_2 \in R_{2,1}$ give

$$W = \mathbb{P}(\mathrm{Span}(f_0, f_1, f_2)) \subseteq \mathbb{P}(R_{2,1}) \simeq \mathbb{P}^5$$

since $R_{2,1} = \mathrm{Span}(x^2z, x^2w, xyz, xyw, y^2z, y^2w)$.

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The multiplication map $R_{2,0} \times R_{0,1} \to R_{2,1}$ gives

$$\mathbb{P}^2 \times \mathbb{P}^1 = \mathbb{P}(R_{2,0}) \times \mathbb{P}(R_{0,1}) \to \mathbb{P}^5 = \mathbb{P}(R_{2,1}).$$

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Key Question: What is $W \cap Y$?

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Thus:

- $W = \mathbb{P}(\mathrm{Span}(f_0, f_1, f_2)) \subseteq \mathbb{P}(R_{2,1}) \simeq \mathbb{P}^5$.
- Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \simeq Y \subseteq \mathbb{P}^5$.
- f_i don't vanish simultaneously on $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem: Either

- Generic: $W \cap Y$ is finite and $\mathrm{Syz}(f)_{2,3} = 0$, or
- Special: $W \cap Y$ is a smooth conic in $W \simeq \mathbb{P}^2$, and $\mathrm{Syz}(f)_{2,3} \neq 0$.

This tells us when the question marks exist.

Generators of $\mathrm{Syz}(f)$

Generic: Six minimal generators:

- One in bidegree $(6, 1)$
- Three in bidegree $(4, 2)$ (Koszul)
- Two in bidegree $(3, 3)$.

Special: Five minimal generators:

- One in bidegree $(6, 1)$
- Three in bidegree $(4, 2)$ (Koszul)
- One in bidegree $(2, 3)$.

Two Resolutions

Generic:

$$\begin{array}{ccccccccc} & & R(-6,-1) & & & & & & \\ & & \bigoplus & & & & & & \\ \cdots & \to & R(-4,-2)^3 & \to & R(-2,-1)^3 & \to & I & \to & 0 \\ & & \bigoplus & & & & & & \\ & & R(-3,-3)^2 & & & & & & \end{array}$$

Special:

$$\begin{array}{ccccccccc} & & R(-6,-1) & & & & & & \\ & & \bigoplus & & & & & & \\ \cdots & \to & R(-4,-2)^3 & \to & R(-2,-1)^3 & \to & I & \to & 0 \\ & & \bigoplus & & & & & & \\ & & R(-2,-3) & & & & & & \end{array}$$

Finding Resolutions

To find the minimal free resolution of the ideal $I = \langle f_0, f_1, f_2 \rangle$, we will use the following tools:

- Koszul Resolutions
- Hilbert-Burch Resolutions
- Mapping Cones

$$\begin{array}{ccccccccc} & & F_\bullet & \dashrightarrow & G_\bullet & & MC_\bullet & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S/(\mathcal{I} : f) & \xrightarrow{f} & S/\mathcal{I} & \longrightarrow & S/(\mathcal{I} + \langle f \rangle) & \to & 0 \end{array}$$

Generic Strategy

First decompose $I = \langle f_0, f_1, f_2 \rangle$ into

$$I = \underbrace{\langle f_0, f_1 \rangle}_{K} + \langle f_2 \rangle$$

where K has a Koszul resolution. Then prove

$$K : f_2 = \langle f_0, f_1, k_1, k_2, g \rangle = \underbrace{\langle f_1, f_2, k_1, k_2 \rangle}_{H} + \langle g \rangle$$

where H has a Hilbert-Burch resolution. Finally,

$$H : g = \langle z, w \rangle.$$

Then apply mapping cone twice.

More on $K : f_2$

Using $K = \langle f_0, f_1 \rangle$, we obtain

$$K : f_2 = \{u \in R \mid u f_2 = A f_0 + B f_1,\ A, B \in R\}.$$

Since we know generators of $\mathrm{Syz}(f)$, we get

$$K : f_2 = \langle \underbrace{f_0, f_1}_{(2,1)}, \underbrace{k_1, k_2}_{(1,2)}, \underbrace{g}_{(4,0)} \rangle$$

since the generators of $\mathrm{Syz}(f)$ have bidegrees $(4, 2), (3, 3), (6, 1)$ and we shift by $(2, 1)$.

More on k_1, k_2

We also have explicit formulas for k_1, k_2. If

$$f_0 = C_0x^2 + D_0xy + E_0y^2$$
$$f_1 = C_1x^2 + D_1xy + E_1y^2$$

(C_i, D_i, E_i linear in z, w), then

$$k_1 = \det\begin{bmatrix} C_0x + D_0y & E_0 \\ C_1x + D_1y & E_1 \end{bmatrix},\ k_2 = \det\begin{bmatrix} C_0 & D_0x + E_0y \\ C_1 & D_1x + E_1y \end{bmatrix}.$$

These are examples of Sylvester forms.

More on H

It follows that $H = \langle f_0, f_1, k_1, k_2 \rangle$ is generated by the 3×3 minors of the 4×3 matrix:

$$\begin{bmatrix} E_1 & -D_1 & C_1 \\ E_0 & -D_0 & C_0 \\ x & y & 0 \\ 0 & x & y \end{bmatrix}$$

Hence K has a Hilbert-Burch Resolution.

Putting this all together, we can resolve $I = \langle f_0, f_1, f_2 \rangle$ in the generic case.

Generic Resolution

$$0 \longrightarrow R(-6,-3) \longrightarrow \begin{array}{c} R(-4,-3)^3 \\ \bigoplus \\ R(-6,-2)^2 \end{array} \longrightarrow$$

$$\begin{array}{c} R(-6,-1) \\ \bigoplus \\ R(-4,-2)^3 \\ \bigoplus \\ R(-3,-3)^2 \end{array} \longrightarrow R(-2,-1)^3 \longrightarrow I \longrightarrow 0$$

Conclusion

- The special case can be handled using the same methods.
- Full details appear in the book chapter A Case Study in Bigraded Commutative Algebra by Cox, Dickenstein and Schenck, in *Syzygies and Hilbert Functions*, I. Peeva ed., Chapman & Hall/CRC, 2007.

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- Thank You!