# $h$-VECTORS OF GORENSTEIN POLYTOPES

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ABSTRACT. We show that the Ehrhart $h$-vector of an integer Gorenstein polytope with a unimodular triangulation satisfies McMullen's $g$-theorem; in particular it is unimodal. This result generalizes a recent theorem of Athanasiadis (conjectured by Stanley) for compressed polytopes. It is derived from a more general theorem on Gorenstein affine normal monoids $M$: one can factor $K[M]$ ($K$ a field) by a "long" regular sequence in such a way that the quotient is still a normal affine monoid algebra. In the case of a polytopal Gorenstein normal monoid $E(P)$, this technique reduces all questions about the Ehrhart $h$-vector to a normal Gorenstein polytope $Q$ with exactly one interior lattice point. (These are the normal ones among the reflexive polytopes considered in connection with mirror symmetry.) If $P$ has a unimodular triangulation, then it follows readily that the Ehrhart $h$-vector of $P$ coincides with the $h$-vector of the boundary complex of a simplicial polytope, and the $g$-theorem applies.

## 1. INTRODUCTION

Let $P \subset \mathbb{R}^{n-1}$ be an integral convex polytope and consider the *Ehrhart-function* given by $E(P,m) = |\{z \in \mathbb{Z}^{n-1} : \frac{z}{m} \in P\}|$ for $m > 0$ and $E(P,0) = 1$. It is well-known that $E(P,m)$ is a polynomial in $m$ of degree $\dim(P)$ and the corresponding *Ehrhart-series* $E_P(t) = \sum_{m\in\mathbb{N}} E(P,m)t^m$ is a rational function

$$E_P(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{\dim(P)+1}}.$$

We call $h(P) = (h_0,\ldots,h_d)$ (where $h_d \neq 0$) the *$h$-vector* of $P$. This vector was intensively studied in the last decades (e.g. see [5] or [9]). In particular, the following questions are of interest:

(i) For which polytopes is $h(P)$ *symmetric*, i.e. $h_i = h_{d-i}$ for all $i$?
(ii) For which polytopes is $h(P)$ *unimodal*, i. e. there exists a natural number $t$ such that $h_0 \leq h_1 \leq \cdots \leq h_t \geq h_{t+1} \geq \cdots \geq h_d$?

Let us sketch Stanley's approach to Ehrhart functions via commutative algebra. The results we are referring to can be found in [5] or [9]. The Ehrhart function of $P$ can be interpreted as the Hilbert function of an affine monoid algebra $K[E(P)]$ (with coefficients from an arbitrary field $K$). Namely, one considers the cone $C(P)$ generated by $P \times \{1\}$ in $\mathbb{R}^n$, and sets $E(P) = C(P) \cap \mathbb{Z}^n$. The algebra $K[E(P)]$ is graded in such a way that the degree of a monomial $x \in E(P)$ is its last coordinate, and so the Hilbert function of $K[E(P)]$ coincides with the Ehrhart function of $P$. Since $P$ is integral, $K[E(P)]$ is a finite module over its subalgebra generated in degree 1.

However, in general $E(P)$ is not generated by its degree 1 elements. If it is, then we say that $P$ is *normal*, and simplify our notation by setting $K[P] = K[E(P)]$.

The monoid $E(P)$ is always normal, and by a theorem of Hochster, $K[E(P)]$ is a Cohen-Macaulay algebra. It follows that $h_i \geq 0$ for all $i = 1, \dots, d$. Using Stanley's Hilbert series characterization of the Gorenstein rings among the Cohen-Macaulay domains, one sees that $h(P)$ is symmetric if and only if $K[E(P)]$ is a Gorenstein ring. In terms of the monoid $E(P)$, the Gorenstein property has a simple interpretation: it holds if and only if $E(P) \cap \operatorname{int} C(P)$ is of the form $x + E(P)$ for some $x \in E(P)$. This follows from the description of the canonical module of normal affine monoid algebras by Danilov and Stanley.

It was conjectured by Stanley that question (ii) has a positive answer for the *Birkhoff polytope* $P$, whose points are the real doubly stochastic $n \times n$ matrices and for which $E(P)$ encodes the *magic squares*. This long standing conjecture was recently proved by Athanasiadis [1]. (That $P$ is normal and $K[P]$ is Gorenstein in this case is easy to see.)

Questions (i) and (ii) can be asked similarly for the combinatorial $h$-vector $h(\Delta(Q))$ of the boundary complex $\Delta(Q)$ of a simplicial polytope $Q$, and both have a positive answer. The Dehn-Sommerville equations express the symmetry, while unimodality follows from McMullen's famous $g$-theorem (proved by Stanley [8]): the vector $(1, h_1 - h_0, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an $M$-sequence, i. e. it represents the Hilbert function of a graded artinian $K$-algebra that is generated by its degree 1 elements. In particular, its entries are nonnegative, and so the $h$-vector is unimodal.

Athanasiadis proved Stanley's conjecture for the Birkhoff polytope $P$ by showing that there exists a simplicial polytope $Q$ with $h(\Delta(Q)) = h(P)$. More generally, his theorem applies to compressed polytopes i, e. integer polytopes all of whose pulling triangulations are unimodular. (The Birkhoff polytope is compressed [7, 9].) In this note we generalize Athanasiadis' theorem as follows:

**Theorem 1.** *Let $P$ be an integral polytope such that $P$ has a unimodular triangulation and $K[P]$ is Gorenstein. Then the $h$-vector of $P$ satisfies the inequalities $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor d/2 \rfloor}$. More precisely, the vector $(1, h_1 - h_0, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an $M$-sequence.*

Our strategy of proof is to consider the algebra $K[M]$ of a normal affine monoid $M$ for which $K[M]$ is Gorenstein. We relate the Hilbert series of $K[M]$ to that of a simpler affine monoid algebra $K[N]$ which we get by factoring out a suitable regular sequence of $K[M]$. In the situation of an algebra $K[P]$ for a normal polytope $P$, the regular sequence is of degree 1, and we obtain a normal reflexive polytope such that $h(P) = h(Q)$. (However, note that Mustaţa and Payne [6] have given an example of a nonormal reflexive polytope with a nonunimodal $h$-vector.) If $P$ has even a unimodular triangulation, we can find a simplicial polytope $P'$ such that the $h$-vector of the boundary complex of $P'$ coincides with the one of $K[P]$. Then it only remains to apply the $g$-theorem to $P'$.

As a side effect we show that the toric ideal of a Gorenstein polytope with a square-free initial ideal has also a Gorenstein square-free initial ideal.

For notions and results related to commutative algebra we refer to Bruns–Herzog [5] and Stanley [9]. For details on convex geometry we refer to the books of Bruns and Gubeladze [3] (in preparation) and Ziegler [11].

## 2. GORENSTEIN MONOID ALGEBRAS

We fix a field $K$ for the rest of the paper. Given a positive affine monoid $M \subseteq \mathbb{Z}^n$ we consider the rational cone $\mathrm{cn}(M) \subseteq \mathbb{R}^n$ generated by $M$. If $\mathrm{cn}(M) = \bigcap_{i=1}^s H_{\sigma_i}^+$ is the irredundant intersection of rational half-spaces, then each $\sigma_i$ is unique up to a nonnegative factor. There is a unique multiple with coprime integral coefficients, and we call this choice of $\sigma_i$, $i = 1,\dots,s$, the *support forms* of $M$. The map

$$\sigma\colon M \to \mathbb{Z}^s, \quad a \mapsto (\sigma_1(a),\dots,\sigma_s(a)),$$

is injective because $M$ is positive. It is called the *standard embedding* of $M$. It can be extended to $\mathrm{gp}(M)$, and we denote the extension also by $\sigma$.

**Lemma 2.** *Let $M \subseteq \mathbb{Z}^n$ be a positive normal affine monoid with $\mathrm{gp}(M) = \mathbb{Z}^n$ and $R = K[M]$. Let $\sigma_1,\dots,\sigma_s$ be the support forms and $\sigma\colon \mathbb{Z}^n \to \mathbb{Z}^s$ the standard embedding of $M$. Then:*

(i) *The $\mathbb{Z}^n$-graded canonical module $\omega_R$ is the ideal of $R$ generated by all $X^z$ for $z \in \mathrm{int}(M)$.*
(ii) *$R$ is Gorenstein if and only if there exists a (necessarily unique) $y \in \mathrm{int}(M)$ such that $\mathrm{int}(M) = y + M$, and therefore $\omega_R = (X^y)$.*
(iii) *$R$ is Gorenstein if and only if there exists a (necessarily unique) $y \in \mathrm{int}(M)$ such that $\sigma(y) = (1,\dots,1)$.*

*Proof.* (i) and (ii) are well-known results of Stanley and Danilov. A proof can be found in [5].

(iii): Assume that $R$ is Gorenstein. By (ii) there exists $y \in \mathrm{int}(M)$ such that $\omega_R = (X^y)$. We have that $\sigma_i(y) > 0$ for $i = 1,\dots,s$ since $y \in \mathrm{int}(M)$. Fix $i$ and choose $z \in \mathrm{int}(M)$ with $\sigma_i(z) = 1$. Such an element $z$ can be found for the following reason. There exists an element $z' \in M$ such that $\sigma_i(y) = 0$ and $\sigma_j(y) > 0$ for $j \neq i$. Furthermore there exists $z'' \in \mathbb{Z}^n$ such that $\sigma_i(z'') = 1$ by the choice of $\sigma_i$. For $r \gg 0$ the element $z = rz' + z'' \in \mathrm{int}(M)$ will do the job.

Now $z - y \in M$ and thus $\sigma_i(z-y) \geq 0$. Hence $\sigma_i(y) \leq 1$ and therefore $\sigma_i(y) = 1$. This shows that $\sigma(y) = (1,\dots,1)$.

Conversely, if there exists $y \in \mathrm{int}(M)$ such that $\sigma(y) = (1,\dots,1)$ then it is easy to see that $\mathrm{int}(M) = y + M$.

In each case the uniqueness of $y$ follows from the positivity of $M$. □

Let $M \subseteq \mathbb{Z}^n$ be a positive affine monoid. It is well-known that $M$ has only finitely many irreducible elements which form the unique minimal system of generators of $M$. We call the collection of these elements the *Hilbert basis* of $M$. The following is our main result for monoid algebras.

**Theorem 3.** *Let $M \subseteq \mathbb{Z}^n$ be a positive normal affine monoid and assume that $R = K[M]$ is Gorenstein. Let $y_1,\dots,y_m \in \mathrm{Hilb}(M)$ such that $\omega_R = (X^{y_1+\dots+y_m})$ is the $\mathbb{Z}^n$-graded canonical module of $R$. Then:*

(i) *$S = R/(X^{y_1} - X^{y_2},\dots,X^{y_{m-1}} - X^{y_m})$ is isomorphic to a Gorenstein normal affine monoid algebra $K[N]$.*

(ii) *The corresponding multi-graded canonical module $\omega_S$ is generated by the residue class of $X^{y_1}$.*

*Proof.* Replacing $\mathbb{Z}^n$ by $\operatorname{gp}(M)$ if necessary we may assume that $\operatorname{gp}(M) = \mathbb{Z}^n$. Let $\sigma_1, \dots, \sigma_s$ be the support forms of $M$ and $\sigma\colon \operatorname{gp}(M) \to \mathbb{Z}^s$ the standard embedding. By Lemma 2 we have $\sigma(y_1 + \cdots + y_m) = (1, \dots, 1)$ and thus the $\sigma(y_i)$ are $0/1$-vectors with disjoint support. After arranging the facets in a suitable order we may assume that there exist natural numbers $k_i$, $0 = k_0 < k_1 < \cdots < k_m = s$, such that

$$\sigma_j(y_i) = \begin{cases} 1 & \text{if } j \in \{k_{i-1}+1, \dots, k_i\}, \\ 0 & \text{else.} \end{cases}$$

It is easy to see that both sets of elements

$$\{\sigma(y_1), \dots, \sigma(y_m)\} \subseteq \mathbb{Z}^s$$

and

$$\{\sigma(y_1) - \sigma(y_2), \dots, \sigma(y_{m-1}) - \sigma(y_m)\} \subseteq \mathbb{Z}^s$$

respectively are part of a basis for $\mathbb{Z}^s$. Thus the groups generated by

$$y_1, \dots, y_m \quad \text{and} \quad y_1 - y_2, \dots, y_{m-1} - y_m$$

respectively are direct summand of $\mathbb{Z}^n = \operatorname{gp}(M)$, because $\sigma$ is injective. Below we will use these facts several times.

Next we define a subfan $\Gamma$ of the face lattice of $\operatorname{cn}(M)$. In $\Gamma$ we collect all subfaces of those faces $F$ of $\operatorname{cn}(M)$ for which there exist natural numbers

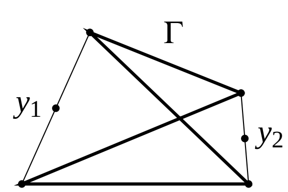
$$i_1, \dots, i_m \quad \text{with } 1 \le i_1 \le k_1, \dots, k_{m-1}+1 \le i_m \le k_m$$

and $F$ has the defining hyperplane

$$H_{i_1,\dots,i_m} = \{z \in \mathbb{R}^n : \sigma_{i_1}(z) + \cdots + \sigma_{i_m}(z) = 0\}.$$

In other words, if $F_i$ is the facet of $\operatorname{cn}(M)$ defined by $\sigma_i$ for $i = 1, \dots, s$, then $F = \bigcap_{j=1}^m F_{i_j}$. Note that $\Gamma$ consists exactly of those faces of $\operatorname{cn}(M)$ that do not contain any of $y_1, \dots, y_m$.

For example, if $K[M] = K[P]$ for the join $P$ of two line segments of length 2 (suitably embedded) and $y_1$ and $y_2$ are the two midpoints (the only possible choice in this case), then $\Gamma$ consists of the cones through the cycle formed by the non-broken edges, as shown in the figure.




Since $M$ is normal in $\operatorname{gp}(M) = \mathbb{Z}^n$, there exists a unimodular triangulation of $\operatorname{cn}(M)$ such that each cone in this triangulation is a unimodular simplicial cone generated by elements of $M$. (See [3, Section 2.D] for a proof of this well-known result.) Restricting this triangulation to $\Gamma$ we obtain a unimodular triangulation $\Sigma$ of $\Gamma$. Now we construct a new unimodular triangulation $\Delta$ of $\operatorname{cn}(M)$ which gives us enough control over its cones. Let

$$\Delta = \Sigma \cup \bigcup_{j=1}^{m} \{\operatorname{cn}(F, y_{i_1}, \ldots, y_{i_j}) : F \in \Sigma,\ 1 \le i_1 < \cdots < i_j \le m\}.$$

We claim that $\Delta$ is a unimodular triangulation of $\operatorname{cn}(M)$. Let $G$ be a face of $\operatorname{cn}(M)$ and choose $x \in \operatorname{int}(G)$. Let

$$\lambda_i = \min\{\sigma_j(x) : j = k_{i-1}+1, \ldots, k_i\} \quad \text{for } i = 1, \ldots, m.$$

Then

$$x' = x - \sum_{i=1}^{m} \lambda_i y_i \in \operatorname{cn}(M) \cap |\Gamma|.$$

Thus there exists an $F \in \Sigma$ such that $x' \in \operatorname{int}(F)$, because $\Sigma$ is a triangulation of $\Gamma$. Hence

$$x \in \operatorname{cn}(F, y_i : \lambda_i > 0),$$

and this cone belongs to $\Delta$. Furthermore $\operatorname{cn}(F, y_i : \lambda_i > 0) \subseteq G$ because of the choice of $x \in \operatorname{int}(G)$. We conclude that every face $G$ of $\operatorname{cn}(M)$ is the union of those faces $\operatorname{cn}(F, y_{i_1}, \ldots, y_{i_j})$ of $\Delta$ which are contained in $G$ and thus $\Delta$ is a subdivision of $\operatorname{cn}(M)$.

It remains to show that the cones in the triangulation $\Delta$ are unimodular and simplicial. Let $F = \operatorname{cn}(M) \cap H_{j_1,\ldots,j_m} \in \Sigma$ be a maximal cone and $\operatorname{cn}(F, y_1, \ldots, y_m) \in \Delta$ be the corresponding maximal cone. Let $v_1, \ldots, v_r$ be the extreme generators of $F$. Since $F$ is unimodular, $v_1, \ldots, v_r$ are linearly independent, and the simplex spanned by these elements together with zero is unimodular. Assume that

$$0 = \sum_{k=1}^{m} \lambda_k y_k + \sum_{l=1}^{r} \mu_l v_l \quad \text{for } \lambda_k, \mu_l \in \mathbb{R}.$$

Applying $\sigma$ we get $0 = \sum_{k=1}^{m} \lambda_k \sigma(y_k) + \sum_{l=1}^{r} \mu_l \sigma(v_l)$. Since $F = \operatorname{cn}(M) \cap H_{j_1,\ldots,j_m}$ and $\sigma(y_{i_k})$ are 0/1-vectors as described above, we conclude that $\lambda_k = 0$ for all $k$. Then, since the $v_l$ are linearly independent, we obtain that $\mu_l = 0$ for all $l$ as well. Hence $\operatorname{cn}(F, y_1, \ldots, y_m)$ is simplicial. Using the standard embedding, it is not hard to see that $\operatorname{cn}(F, y_1, \ldots, y_m)$ is unimodular because a (part of a) basis of $H_{j_1,\ldots,j_m} \cap \mathbb{Z}^n$ can always be extended by $y_1, \ldots, y_m$ to part of a basis for $\mathbb{Z}^n$. We conclude that $\Delta$ is indeed a unimodular triangulation of $\operatorname{cn}(M)$.

Via $\sigma$ the ring $R$ is also $\mathbb{Z}^s$-graded and since the monomials $\sigma(y_i)$ have pairwise disjoint support, the monomials $X^{y_1}, \ldots, X^{y_m}$ form an $R$-sequence. The ring $R$ is an $\mathbb{Z}^n$-graded local ring, since $M$ is positive. Thus $X^{y_1} - X^{y_2}, \ldots, X^{y_{m-1}} - X^{y_m}$ is also an $R$-sequence. (One can alternatively use the fact that $R$ is Cohen–Macaulay.)

Now we define

$$U = \mathbb{Z}^n / (y_1 - y_2, \ldots, y_{m-1} - y_m) \quad \text{and set} \quad N = \varepsilon(M)$$

where $\varepsilon \colon \mathbb{Z}^n \to U$ is the natural projection map. In the following we denote by $\varepsilon$ also the induced map from $\mathbb{R}^n \to \mathbb{R}U$. The map $\varepsilon$ induces a $K$-algebra homomorphism $R \to K[N]$ which factors through $S = R/(X^{y_1} - X^{y_2}, \ldots, X^{y_{m-1}} - X^{y_m})$. We claim that:

(a) $N \subseteq U$ is a positive normal affine monoid;
(b) $S \to K[N]$ is an isomorphism;

(c) $K[N]$ is Gorenstein and the $\mathbb{Z}^n$-graded canonical module $\omega_{K[N]}$ is generated by the residue class of $X^{y_1}$.

Since $N = \varepsilon(M)$, the cone $\operatorname{cn}(N)$ is the union of the unimodular simplicial cones $\varepsilon(G)$ for $G \in \Delta$. Assume that $G = \operatorname{cn}(F, y_{i_1}, \dots, y_{i_j})$ for a unimodular cone $F = \operatorname{cn}(v_1, \dots, v_r) \in \Gamma$ where $v_1, \dots, v_r \subseteq M$ are the extreme generators of $F$. Then

$$G = \operatorname{cn}(\varepsilon(v_1), \dots, \varepsilon(v_r), p) \subseteq \mathbb{R}U \quad \text{or} \quad G = \operatorname{cn}(\varepsilon(v_1), \dots, \varepsilon(v_r)) \subseteq \mathbb{R}U$$

where $p = \varepsilon(y_1) = \cdots = \varepsilon(y_m)$ is the common image of the $y_i$. We know that the elements $v_1, \dots, v_r, y_1, \dots, y_m$ are part of a basis of $\mathbb{Z}^n$ and thus it follows that $\varepsilon(v_1), \dots, \varepsilon(v_r), p$ are part of a basis of $U$. Hence $G$ is unimodular, simplicial and $\operatorname{cn}(N)$ has the unimodular triangulation $\Delta' = \{\varepsilon(G) : G \in \Delta\}$. It follows easily that $N$ is normal (see [3, Section 2.D]). Below we will see that $K[N]$ has a positive grading. Therefore $N$ is positive, and this proves claim (a).

Clearly $S \to K[N]$ is surjective and for (b) it remains to show that this homomorphism is injective. To this end we introduce a new positive grading on $R$, $K[N]$ and $S$ and compare their Hilbert series with respect to this grading. We set

$$\deg(X^a) = k_2 \cdots k_m \sum_{i=1}^{k_1} \sigma_i(a) + \cdots + k_1 \cdots k_{m-1} \sum_{i=k_{m-1}+1}^{k_m} \sigma_i(a)$$

for $a \in \operatorname{gp}(M)$. Note that $\deg(X^{y_i}) = k_1 \cdots k_m$ for $i = 1, \dots, m$. Recall that the maximal cones in the triangulation $\Delta$ of $\operatorname{cn}(M)$ and, therefore, all their intersections contain $y_1, \dots, y_m$. If $C_1, \dots, C_t$ are these maximal cones, then we see via inclusion–exclusion that

$$H_R(t) = \sum_{1 \le i \le t} \sum_{a \in \operatorname{gp}(M) \cap C_i} t^{\deg(X^a)} - \sum_{1 \le i < j \le t} \sum_{a \in \operatorname{gp}(M) \cap C_i \cap C_j} t^{\deg(X^a)} \pm \cdots$$

Hier erstmal $K[N]$

Let $D_1, \dots, D_t$ be the images of $C_1, \dots, C_t$ with respect to $\varepsilon$. Then

$$H_{K[N]}(t) = \sum_{1 \le i \le t} \sum_{a \in \operatorname{gp}(N) \cap D_i} t^{\deg(X^a)} - \sum_{1 \le i < j \le t} \sum_{a \in \operatorname{gp}(M) \cap D_i \cap D_j} t^{\deg(X^a)} \pm \cdots$$

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A comparison of the elements in the unimodular simplicial cones $C_i$ and $D_i$ yields

$$H_{K[N]}(t) = (1 - t^{k_1 \cdots k_m})^{m-1} H_R(t).$$

But the right hand side of the latter equation is exactly the Hilbert series of $S$, because $X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m}$ is a regular sequence of $R$. Hence $H_S(t) = H_{K[N]}(t)$ and therefore $S \cong K[N]$, which is claim (b). Since $S$ is clearly a Gorenstein ring, its isomorphic copy $K[N]$ is Gorenstein, too, and so the first statement in (c) has been proved.

It remains to compute the multi-graded canonical module of $S \cong K[N]$. Since $K[N]$ is Gorenstein, we have to determine the unique lattice point $q$ in $\operatorname{int}(N)$ such that $\operatorname{int}(N) = q + N$ because then $\omega_{K[N]} = (X^q)$. By construction, $q$ must have degree $k_1 \cdots k_m$, and the residue class of $y_1$ in $U$ is an interior point of $\operatorname{cn}(N)$ of that degree. This concludes the proof. $\square$

## 3. GORENSTEIN POLYTOPES

Let $X \subseteq \mathbb{R}^{n-1}$. We set $E(X,m) = |\{z \in \mathbb{Z}^{n-1} : \frac{z}{m} \in P\}|$ and $E(P,0) = 0$. In analogy to the rational function $E_P(t)$ we define

$$E_{\mathrm{int}(P)}(t) = \sum_{m\in\mathbb{N}} E(\mathrm{int}(P),m)t^m \quad \text{and} \quad E_{\partial(P)}(t) = \sum_{m\in\mathbb{N}} E(\partial(P),m)t^m.$$

Observe that $E_{\partial(P)}(t) = E_P(t) - E_{\mathrm{int}(P)}(t)$. In our situation $E_P(t) = H_{K[P]}(t)$ and $E_{\mathrm{int}(P)}(t) = H_{\omega_{K[P]}}(t)$. Hence these series are rational with denominator $(1-t)^{\dim(P)+1}$. Moreover, $E_{\partial(P)}(t) = E_P(t) - E_{\mathrm{int}(P)}(t)$ is rational with denominator $(1-t)^{\dim(P)}$, and it makes sense to consider the $h$-vectors of these series which we denote by $h(\mathrm{int}(P))$ and $h(\partial(P))$. In the following we present variations and corollaries of Theorem 3.

**Corollary 4.** *Let $P$ be an normal integer polytope such that $K[P]$ is Gorenstein. Then there exists a Gorenstein normal integer polytope $Q$ such that* $\mathrm{int}(Q)$ *contains a unique lattice point and*

$$h(P) = h(Q) = h(\partial(Q)).$$

*Proof.* Recall that $R = K[P]$ is the affine monoid ring generated by the positive normal affine monoid $M = E(P) = C \cap \mathbb{Z}^n$ where $C = \mathrm{cn}((p,1) : p \in P)$. Observe that $R$ is $\mathbb{Z}$-graded with respect to the last coordinate and we will use only this grading for the rest of the proof. All irreducible elements of $M$ have degree 1, because $P$ is normal. Since $R$ is Gorenstein, there exists a unique lattice point $y \in M$ such that $\mathrm{int}(M) = y + M$. Choosing irreducible elements $y_1,\dots,y_m \in M$ such that $y = \sum_{i=1}^m y_i$ we are in the situation to apply Theorem 3.

In the proof of the theorem we have constructed the lattice $U = \mathrm{gp}(M)/(y_i - y_{i+1} : i = 1,\dots,m-1)$ and the normal affine lattice monoid $N \subset \mathrm{gp}(M)$ such that $K[N]$ is Gorenstein. The monoid $N$ is also homogeneous with respect to the grading induced by that of $M$ and generated by the degree 1 elements. Thus it is polytopal by [4, Proposition 1.1.3], and $K[N] = K[Q]$ for the polytope $Q$ spanned by the degree 1 elements of $N$. It has also been shown that the canonical module of $K[Q]$ is generated by a degree 1 element, the residue class of $X^{y_1}$, which we denote by $X^p$. Thus $Q$ can have only one interior lattice point, namely $p$. The $h$-polynomial of $K[P]$ and the one of $K[Q]$ coincide since $K[Q] \cong K[P]/(X^{y_i} - X^{y_{i+1}})$ and $X^{y_1} - X^{y_2},\dots,X^{y_{m-1}} - X^{y_m}$ is a regular sequence homogeneous of degree on 1.

It follows from

$$E_{\partial(P)}(t) = E_P(t) - E_{\mathrm{int}(P)}(t) = H_{K[P]}(t) - H_{\omega_{K[P]}}(t) = H_{K[P]}(t) - t \cdot H_{K[P]}(t)$$

that $h(Q) = h(\partial(Q))$. For the last equality we have used the fact that $\omega_{K[P]} = (X^p) \cong R(-1)$ with respect to the considered grading. This concludes the proof. □

The Gorenstein polytopes with an interior lattice point are exactly the *reflexive polytopes* used by Batyrev in the theory of mirror symmetry; see [2]. Therefore the previous corollary reduces all questions about the $h$-vector of normal Gorenstein polytopes to normal reflexive polytopes. However, as shown by Mustaţa and Payne [6], there exist nonnormal reflexive polytopes whose $h$-vector is not unimodal.

If $Q$ is a simplicial polytope, then its boundary complex $\Delta(Q)$ is simplicial, and we can speak of its combinatorial $h$-vector (which one can read as the $h$-vector of the Ehrhart series of the geometric realization of $\Delta(Q)$ in the boundary of a suitable unit simplex.)

**Corollary 5.** *Let $P$ be an integer polytope such that $K[P]$ is Gorenstein and $P$ has a unimodular triangulation. Then there exists a simplicial integer polytope $P'$ such that*

$$h(P) = h(\Delta(P')).$$

*Proof.* Polytopes with a unimodular triangulation are normal. So we can proceed as in the proof of Corollary 4 and use the same notation. The only change is that we start with the given unimodular triangulation $\Sigma$ of $P$. It induces a unimodular triangulation of $\operatorname{cn}((p,1) : p \in P)$ that can be used in the proof of Theorem 3. Thus the simplicial cones in that triangulation have generators of degree one. This induces a unimodular triangulation of $\operatorname{cn}(N)$ with generators of degree one and thus a unimodular triangulation $\Sigma'$ of the (normal) integer polytope $Q$. Moreover, $K[Q]$ is Gorenstein, $h(P) = h(Q) = h(\partial(Q))$ and $\operatorname{int}(Q)$ contains a unique interior lattice point $p$. It only remains to construct the simplicial polytope $P'$.

We project the vertices of the triangulation of $Q$ on a sphere around $p$ inside $Q$ and consider their convex hull $P'$. It is a simplicial polytope whose boundary is combinatorially equivalent $\Sigma'$. Since $\Sigma'$ is unimodular the Ehrhart $h$-vector $h(\partial Q)$ coincides with the combinatorial $h$-vector of $\Sigma'$, and hence with that of $\Delta(P')$. We obtain

$$h(\partial(Q)) = h(\Delta(P')),$$

as desired. □

The assumptions of Corollary 5 appear at several places in algebraic combinatorics as has been pointed out in [1]. Theorem 1 follows immediately by the $g$-theorem.

We conclude by drawing a consequence for the toric ideal $I_P$ of $P$. It defines the algebra $K[P]$ in the form $K[P] = S/I_P$ where $S$ is a polynomial ring on the integral points of $P$. "Generic" weights on $S$ induce on the one side regular triangulations $\Sigma$ of $P$ and on the other side weight orders $>$ on $S$; see Sturmfels [10] for the details. The initial ideal of $I_P$ with respect to $>$ is then a monomial ideal $J$. By part of the main theorem of [10], $J$ is square-free if and only if the triangulation is unimodular. In this case $S/J$ is the Stanley-Reisner ring of $\Sigma$, understood as an abstract simplicial complex. If in the situation of Corollary 5 the unimodular triangulation of $P$ regular, then one can show that the induced triangulation constructed above is also regular (see [3, Section 1.F]) and apply the result of Sturmfels to find a new initial ideal $J'$. By construction, its underlying triangulation is combinatorially the join of $\Delta(P')$ (as in Corollary 5) and the simplex spanned by $y_1,\dots,y_m$ (as in Theorem 3). So the indeterminates of $S$ corresponding to $y_1,\dots,y_m$ form a regular sequence modulo $J'$, and we obtain

**Corollary 6.** *Let $P$ be an integer Gorenstein polytope with a regular unimodular triangulation. Then the toric ideal $I_P$ has a square-free initial ideal defining a Gorenstein ring.*

The corollary answers a question of Conca and Welker, and the methods of this note were originally designed for its solution.

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