An example of using Representation Theory to find a resolution

Lecture two
August 31, 2005

Section 1. Irreducible Representations.

Let V be a finite dimensional vector space over the field K of characteristic zero. Last week we learned that for every partition λ, there is an irreducible $GL(V)$-representation $L_\lambda(V)$. This object was described in a coordinate free manner. However, I can tell you a basis for it. I told you that $L_\lambda(V)$ is the image of a map

$$\bigwedge^{\lambda_1} V \otimes \ldots \otimes \bigwedge^{\lambda_\ell} V \to \operatorname{Sym}_{\lambda'_1} V \otimes \ldots \otimes \operatorname{Sym}_{\lambda'_{\text{last}}} V.$$

I will express my basis as elements of $\frac{\text{The left hand side}}{\text{kernel}}$. If $v_1, \ldots, v_n$ is a basis for V, then one basis for $L_\lambda V$ is

$$\left\{ \begin{array}{l} \quad v_{i_{1,1}} \wedge v_{i_{1,2}} \wedge \ldots \wedge \qquad\quad v_{i_{1,\lambda_1}} \\ \otimes v_{i_{2,1}} \wedge v_{i_{2,2}} \wedge \ldots \wedge v_{i_{2,\lambda_2}} \\ \otimes \\ \vdots \\ \otimes v_{i_{\ell,1}} \wedge v_{i_{\ell,2}} \wedge \ldots \end{array} \right\}$$

where the indices satisfy

$$1 \le i_{r,1} < i_{r,2} < \cdots < i_{r,\lambda_r} \le n$$

in each row and

$$\begin{array}{c} i_{1,c} \\ \wedge| \\ i_{2,c} \\ \wedge| \\ i_{3,c} \\ \wedge| \\ \vdots \end{array}$$

in each column.

Section 2. A family of complexes associated to a generic map $\varphi\colon F \to G$.

Let F and G be vector spaces over the old field K of dimension f and g respectively; let $R = K[\{\varphi_{i,j}\}]$ be the polynomial ring in the $g \times f$ variables $\varphi_{i,j}$; and let $\varphi\colon F \otimes_K R \to G \otimes_K R$ be the R-module homomorphism given by multiplication by the matrix $[\varphi_{i,j}]$. (A useful, coordinate-free, description looks like "let $R = \operatorname{Sym}_K(F \otimes G^*)$ and $\varphi\colon F \otimes_K R \to G \otimes_K R$ be the natural R-module homomorphism".)

For each partition $\nu = (\nu_1, \dots, \nu_{g-1})$, I will tell give you a collection t_ν of free R-modules and R-module homomorphisms:

$$\to t_{\nu,k} \to t_{\nu,k-1} \to \dots\ .$$

Eventually, I will show you that each t_ν is a complex, and is acyclic. It is easy to see that $H_0(t_\nu)$ is a module over $R/I_g(\varphi)$. When we look at the complexes we will see that the length of t_ν is $f-g+1$ when $\nu_1 \le f-g+1$; so in these cases $H_0(t_\nu)$ is a maximal Cohen Macaulay $R/I_g(\varphi)$-module (and a perfect R-module).

The notation. Given the partition $\nu = (\nu_1, \dots, \nu_{g-1})$ and an integer k.

- Find i with $\nu_i \ge k > \nu_{i+1}$. Let

$$p(\nu,k) = (\nu_1, \dots, \nu_i, k, \nu_{i+1}+1, \dots, \nu_{g-1}+1).$$

Let

$$N(\nu,k) = |p(\nu,k)| - |\nu| = k+g-1-i.$$

The modules. Let $t_{\nu,k} = \bigwedge^{N(\nu,k)} F \otimes_K L_{p(\nu,k)'} G^* \otimes_K R$.

Examples.

- If $g=4$, then $t_{(0,0,0)}$ is

$$\cdots \to \bigwedge^6 F \otimes \underbrace{L_{(3,1,1,1)'} G^*}_{D_2 G^* \otimes \bigwedge^4 G^*} \otimes R \to \bigwedge^5 F \otimes \underbrace{L_{(2,1,1,1)'} G^*}_{D_1 G^* \otimes \bigwedge^4 G^*} \otimes R$$

$$\to \bigwedge^4 F \otimes \underbrace{L_{(1,1,1,1)'} G^*}_{\bigwedge^4 G^*} \otimes R \to \bigwedge^0 F \otimes \underbrace{L_{(0,0,0,0)'} G^*}_{K} \otimes R$$

(This is the Eagon-Northcott complex which resolves $R/I_4(\varphi)$.)

• If $g=4$, then $t_{(1,1,1)}$ is

$$\cdots \to \bigwedge^6 F \otimes \underbrace{L_{(3,2,2,2)'}G^*}_{D_1 G^* \otimes \bigwedge^4 G^* \otimes \bigwedge^4 G^*} \otimes R \to \bigwedge^5 F \otimes \underbrace{L_{(2,2,2,2)'}G^*}_{\bigwedge^4 G^* \otimes \bigwedge^4 G^*} \otimes R$$

$$\to \bigwedge^1 F \otimes \underbrace{L_{(1,1,1,1)'}G^*}_{\bigwedge^4 G^*} \otimes R \to \bigwedge^0 F \otimes \underbrace{L_{(1,1,1,0)'}G^*}_{\operatorname{Sym}_1 G \otimes \bigwedge^4 G^*} \otimes R$$

(This is the "Buchsbaum-Rim" complex which resolves the cokernel of the generic map $F \xrightarrow{\varphi} G$.)

• If $g=4$, then $t_{(2,1,0)}$ is

$$\cdots \to \bigwedge^7 F \otimes L_{(4,3,2,1)'}G^* \otimes R \to \bigwedge^6 F \otimes L_{(3,3,2,1)'}G^* \otimes R \to \bigwedge^4 F \otimes L_{(2,2,2,1)'}G^* \otimes R$$

$$\to \bigwedge^2 F \otimes L_{(2,1,1,1)'}G^* \otimes R \to \bigwedge^0 F \otimes L_{(2,1,0,0)'}G^* \otimes R.$$

I included this example merely to point out that the family of complexes under consideration is much larger than the family of Eagon-Northcott complexes. One can tell the degree of the differential by looking at the difference in the power of $\bigwedge F$. The present example has three matrices of quadratic maps before linear maps finally appear. An Eagon-Northcott complex (see for example Appendix A2.6 in Eisenbud) has linear maps in every position except one.

The differentials. I will tell you the differential

$$t_{\nu,k} \to t_{\nu,k-1}. \tag{**}$$

The partition ν is $(\nu_1,\dots,\nu_i,\underbrace{k-1,\dots,k-1}_{m},\nu_j,\dots,\nu_{g-1})$, with $\nu_i \ge k$ and $k-2 \ge \nu_j$. In this case,

$$p(\nu,k) = (\alpha, k^{m+1}, \beta) \quad \text{and} \quad p(\nu,k-1) = (\alpha,(k-1)^{m+1},\beta)$$

for $\alpha = (\nu_1,\dots,\nu_i)$ and $\beta = (\nu_j+1,\dots,\nu_{g-1}+1)$. When K has characteristic zero, Representation Theory establishes the existence of a map

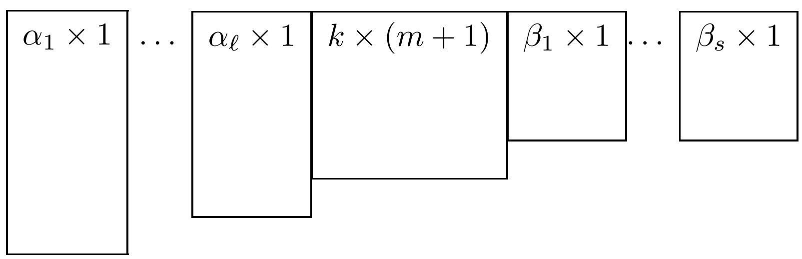
$$L_{(\alpha,k^{m+1},\beta)'}G^* \xrightarrow{\text{RT}} L_{(1^{m+1})'}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*. \tag{***}$$

Of course, $L_{(1^{m+1})'}G^*$ is a very odd way of writing $\bigwedge^{m+1} G^*$. The map (**) is

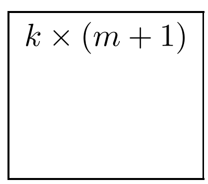
$$t_{\nu;k} = \bigwedge^{N(\nu;k)} F \otimes L_{(\alpha,k^{m+1},\beta)'}G^* \otimes R \xrightarrow{\text{RT}} \bigwedge^{N(\nu;k)} F \otimes \bigwedge^{m+1} G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*$$
$$\xrightarrow{\bigwedge^{m+1}\varphi^*} \bigwedge^{N(\nu,k)} F \otimes \bigwedge^{m+1} F^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^* \xrightarrow{\text{module action}}$$
$$\bigwedge^{N(\nu;k)-(m+1)} F^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^* = t_{\nu,k-1}$$

(In other words, use Representation Theory to pull $m+1$ boxes from row k and then do the "obvious map" involving the $m+1 \times m+1$ minors of φ.)

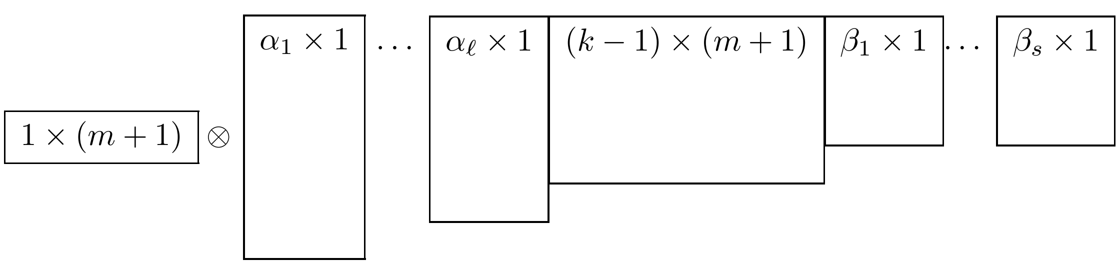
The picture that goes with (***) is:



The Representation Theory allows us to move the bottom row from



and get



The reason that Representation Theory gives the map (***): The module $L_{m+1}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*$ is a direct sum of irreducible representations of $\mathrm{GL}(G^*)$. The Littlewood-Richardson rule tells us that exactly one copy of $L_{(\alpha,k^{m+1},\beta)'}G^*$ appears in this direct sum decomposition. So (up to scalar multiple from K), there is exactly one coordinate-free map (***). It is possible to write down exactly what (***) does, but it is not a very pretty answer!

Section 3. Each t_ν is a complex.

The Littlewood-Richardson Rule. *If λ and μ are partitions then*

$$L_\lambda V \otimes L_\mu V = \bigoplus \mathrm{LR}(\lambda,\mu;\nu) L_\nu V$$

where the sum is taken over all partitions ν with $|\nu| = |\lambda| + |\mu|$, and the Littlewood-Richardson coefficient $\mathrm{LR}(\lambda,\mu;\nu)$ is is calculated according to the following description. Draw ν, remove μ. Fill in the resulting picture using λ_1 ones, λ_2 twos, etc. You must have your rows WEAKLY increasing and your columns STRICTLY increasing. The word that you form using the Macdonald convention (right to left top to bottom) must be a lattice permutation meaning $w = a_1a_2\ldots a_N$ in the symbols $1,2,\ldots,n$ is a lattice permutation if for $1 \le r \le N$ and $1 \le i \le n-1$, the number of occurrences of the symbol i in $a_1a_2\ldots a_r$ is not less than the number of occurrences of $i+1$.

Example. Let us calculate the LR coefficient for $L_{(\alpha,k^{m+1},\beta)'}G^*$ in

$$L_{m+1}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*.$$

We draw $(\alpha, k^{m+1}, \beta)'$ and remove $(\alpha,(k-1)^{m+1},\beta)'$. This leaves

$$\boxed{\text{a one } m+1 \text{ row of boxes}}.$$

We must fill these boxes in using $m+1$ ones. There is one way to do this. This unique way is weakly increasing in the rows and the word is okay! **Thus, there is exactly one non-zero $\mathrm{GL}(V)$-module homomorphism**

$$L_{(\alpha,k^{m+1},\beta)'}G^* \to L_{m+1}G^* \otimes L_{(\alpha,(k-1)^{m+1},\beta)'}G^*$$

(up to multiplication by a scalar).

Calculation. Now we show that each t_ν is a complex. That is, we show that the composition

$$(\dagger) \qquad t_{\nu,k} \to t_{\nu,k-1} \to t_{\nu,k-2}$$

is zero. Write $\nu = (\nu_1, \dots, \nu_i, (k-1)^a, (k-2)^b, \nu_j, \dots, \nu_{g-1})$, with $\nu_i \geq k$ and $k-3 \geq \nu_j$. Let $\alpha = (\nu_1, \dots, \nu_i)$ and $\beta = (\nu_j + 1, \dots, \nu_{g-1} + 1)$. We see that

$$p(\nu; k) = (\alpha, k^{a+1}, (k-1)^b, \beta), \quad p(\nu; k-1) = (\alpha, (k-1)^{a+b+1}, \beta)$$

and

$$p(\nu; k-2) = (\alpha, (k-1)^a, (k-2)^{b+1}, \beta).$$

The composition $(\dagger)$ is

$$t_{\nu,k} = \bigwedge^{N(\nu;k)} F \otimes L_{(\alpha,k^{a+1},(k-1)^b,\beta)'} G^* \otimes R \xrightarrow{\text{RT}}$$

$$\bigwedge^{N(\nu;k)} F \otimes \bigwedge^{a+1} G^* \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'} G^* \otimes R \xrightarrow{\varphi^*}$$

$$\bigwedge^{N(\nu;k)} F \otimes \bigwedge^{a+1} F^* \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'} G^* \otimes R \xrightarrow{\text{MA}}$$

$$\bigwedge^{N(\nu;k)-(a+1)} F \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'} G^* \otimes R \xrightarrow{\text{RT}}$$

$$\bigwedge^{N(\nu;k)-(a+1)} F \otimes \bigwedge^{b+1} G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'} G^* \otimes R \xrightarrow{\varphi^*}$$

$$\bigwedge^{N(\nu;k)-(a+1)} F \otimes \bigwedge^{b+1} F^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'} G^* \otimes R \xrightarrow{\text{MA}}$$

$$\bigwedge^{N(\nu;k)-(a+1)-(b+1)} F \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'} G^* \otimes R = t_{\nu;k-2}$$

It is legal to do both representation theory maps first, then do both φ^* maps, and then do both module action maps. So we focus on the composition of the Representation Theory maps:

$$L_{(\alpha,k^{a+1},(k-1)^b,\beta)'} G^* \xrightarrow{\text{RT}} \bigwedge^{a+1} G^* \otimes L_{(\alpha,(k-1)^{a+1},(k-1)^b,\beta)'} G^* \xrightarrow{\text{RT}}$$

$$\bigwedge^{a+1} G^* \otimes \bigwedge^{b+1} G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^* \xrightarrow{\text{EM}} \bigwedge^{a+1+b+1} G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)'}G^*$$

(The maps "φ^*" and "Module action" both commute with "Exterior Multiplication".) Well, the Littlewood-Richardson rule tells us that the only $\mathrm{GL}(G^*)$-module map

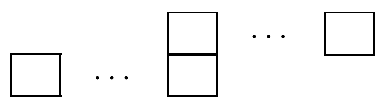
$$(\dagger\dagger) \qquad L_{(\alpha,k^{a+1},(k-1)^b,\beta)}G^* \to L_{a+b+2}G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)}G^*$$

is ZERO, and ($\dagger$) factors through ($\dagger\dagger$); thus, ($\dagger$) is also zero.

To see that the only choice for a $\mathrm{GL}(G^*)$-module map ($\dagger\dagger$) is zero: We compute the LR coefficient for $L_{(\alpha,k^{a+1},(k-1)^b,\beta)}G^*$ in

$$L_{a+b+2}G^* \otimes L_{(\alpha,(k-1)^a,(k-2)^{b+1},\beta)}G^*.$$

We draw the picture for $(\alpha, k^{a+1}, (k-1)^b, \beta)$, remove the picture for $(\alpha, (k-1)^a, (k-2)^{b+1}, \beta)$. We are left with



with $b+1$ boxes in the top row, $a+1$ boxes in the bottom row, and an overlap of one box. There is NO way to fill the picture in using ALL ONES so that the columns are strictly increasing. Thus, **the only coordinate free K-vector space map with domain and range given in ($\dagger\dagger$) is zero** and ($\dagger$) is also zero.

Section 4. Each t_ν is a resolution.