Title: The bi-graded structure of Symmetric Algebras with applications to Rees rings.

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Abstract: Let k be a field, $R = k[x,y]$ a polynomial ring in 2 variables over k, and I a height 2 ideal of R minimally generated by 3 forms, g_1, g_2, g_3 of the same positive degree d. The Hilbert-Burch Theorem guarantees that there is a 3×2 matrix φ, with homogeneous entries from R, so that the signed 2×2 minors of φ are equal to g_1, g_2, and g_3. We arrange φ so that each entry in column i of φ has degree d_i, with $d_1 < d_2$. Let $\mathcal{R}$ be the Rees algebra of I, that is

$$\mathcal{R} = R \oplus I \oplus I^2 \oplus I^3 \oplus \cdots = R[It],$$

$\mathcal{A}$ be the kernel of the natural surjection

$$\mathrm{Sym}(I) \twoheadrightarrow \mathcal{R}$$

from the symmetric algebra of I to the Rees algebra of I and let S and B be the polynomial rings $S = k[T_1, T_2, T_3]$ and $B = R \otimes_k S = k[x, y, T_1, T_2, T_3]$. View B as a bi-graded k-algebra, where x and y have bi-degree $(1,0)$ and each T_i has bi-degree $(0,1)$. We describe the S-module structure of $\mathcal{A}_{(\geq d_1 - 1, *)}$ under the hypothesis that φ has a generalized zero in its first column. This module is free and we identify the bi-degrees of its basis. We also identify the bi-degrees of a minimal generating set of $\mathcal{A}_{(\geq d_1 - 1, *)}$ as an ideal of $\mathrm{Sym}(I)$. Our proof is motivated by a Theorem of Weierstrass and Kronecker which classifies matrices with homogeneous linear entries in two variables – that is, “singular pencils of matrices”. We learned about the Weierstrass-Kronecker Theorem from Gantmacher’s book. When one views this result in a geometric context, that is,

$$[g_1, g_2, g_3] : \mathbb{P}^1 \to \mathcal{C} \subseteq \mathbb{P}^2$$

is a birational parameterization of a plane curve, then Bi-Proj $\mathcal{R}$ is the graph, Γ, of the parameterization of $\mathcal{C}$, the hypothesis concerning the existence of a generalized zero is equivalent to assuming that $\mathcal{C}$ has a singularity of multiplicity d_2, and the ideal $\mathcal{A}_{(\geq d_1 - 1, *)}$ is an approximation of the ideal $\mathcal{A}$ which defines Γ. This is joint work with Claudia Polini (University of Notre Dame) and Bernd Ulrich (Purdue University).