The Generic Hilbert-Burch matrix

AMS meeting at Georgia Southern University

March, 2011

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I have put a copy of this talk on my website.

The talk consists of:

- The set-up
- The questions
- The people involved
- The motivation
- The answer to Question 1 is yes: precise statement and proof.
- The answer to Question 2 is yes: precise statement and proof.
- Extensions
- Other interpretations

The set up:

Let k be a field,

$B = k[x,y]$ be a polynomial ring in two variables over k,

c and d be positive integers with $d = 2c$,

B_c be the vector space of homogeneous forms of degree c in B,

$\mathbb{H}_d$ be the affine space of 3×2 matrices with entries from B_c, (So $\mathbb{H}_d$ is an affine space of dimension $6c+6$.) and

$\mathrm{BalH}_d = \{M \in \mathbb{H}_d \mid \mathrm{ht}(I_2(M)) = 2\}$.

The set up, page 2:

Let $\mathbb{A}_d$ be the affine space $B_d \times B_d \times B_d$, (Each element $\boldsymbol{g}$ of $\mathbb{A}_d$ is a 3-tuple $\boldsymbol{g} = (g_1, g_2, g_3)$, with $g_i \in B_d$. So, $\mathbb{A}_d$ is an affine space of dimension $3d + 3 = 6c + 3$.), and

$\Phi : \mathbb{H}_d \to \mathbb{A}_d$ be the morphism

$$\Phi\left(\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \\ P_5 & P_6 \end{bmatrix}\right) = \left(\begin{vmatrix} P_3 & P_4 \\ P_5 & P_6 \end{vmatrix}, -\begin{vmatrix} P_1 & P_2 \\ P_5 & P_6 \end{vmatrix}, \begin{vmatrix} P_1 & P_2 \\ P_3 & P_4 \end{vmatrix}\right).$$

The set up, page 3:

Notice that: if M is in BalH_d, then M is the Hilbert-Burch matrix for $\Phi(M)$,

$$0 \to B(-3c) \oplus B(-3c) \xrightarrow{M} B(-2c)^3 \xrightarrow{\Phi(M)} B \to B/I_2(M) \to 0$$

is a free resolution of $B/I_2(M)$ and M is a Balanced Hilbert Burch Matrix in the sense that the degrees of the columns of M are as close as possible – namely, the column degrees are equal.

Summary: If $\boldsymbol{g}$ is in $\Phi(\mathrm{BalH}_d)$, then the ideal generated by $\boldsymbol{g}$ has height two and the Hilbert-Burch matrix for the row vector $\boldsymbol{g}$ is Balanced.

The Questions:

Question 1. Can one separate $\Phi(\mathrm{BalH}_d)$ from its complement $\mathbb{A}_d \setminus \Phi(\mathrm{BalH}_d)$, in a polynomial manner. That is, do there exist polynomials $\{F_i\}$ in $6c+3$ variables such that if $\boldsymbol{g}$ is in $\mathbb{A}_d$, then

$$\boldsymbol{g} \in \mathbb{A}_d \setminus \Phi(\mathrm{BalH}_d) \iff F_i(\text{the coefficients of } \boldsymbol{g}) = 0 \text{ for all } i?$$

Question 2. Does the morphism $\Phi : \mathrm{BalH}_d \to \Phi(\mathrm{BalH}_d)$ admit a local section? That is, does there exist an open cover $\{U_j\}$ of $\Phi(\mathrm{BalH}_d)$ such that, for each index j there exists a morphism $\sigma_j : U_j \to \mathbb{A}_d$ with the composition

$$U_j \xrightarrow{\sigma_j} \mathrm{BalH}_d \xrightarrow{\Phi} \mathbb{A}_d$$

equal to the identity map on U_j for all j.

The people involved:

- The original work on the Generic Hilbert-Burch matrix is part of the project with David Cox, Claudia Polini, and Bernd Ulrich. Today's talk is part of section 5 of "A study of singularities on rational curves via syzygies", which we recently posted on the arXiv.

- Very recently, I asked Brett Barwick to explore various questions about Generic Hilbert-Burch matrices. The "extensions" part of the talk is Brett's work.

The Motivation:

David, Claudia, Bernd, and I are in the business of studying singularities on rational plane curves. We fix a parameterization $\boldsymbol{g}$ for the curve and we use information obtained from the Hilbert-Burch matrix for $\boldsymbol{g}$ to describe the singularities of the curve.

We have results that say "When the coefficients of the Hilbert-Burch matrix satisfy all of these polynomials; but not all of those polynomials, then the singularities xxx."

The coefficients of the parameterization are more natural as data than the coefficients of the Hilbert-Burch matrix.

The Generic Hilbert-Burch matrix allows us to express our results in terms of the more natural data the coefficients of the parameterization.

The Answer to both questions is YES

The precise answer to Question 1. If $\boldsymbol{g} = (g_1, g_2, g_3)$ is in $\mathbb{A}_d$ with

$$g_j = z_{0,j}x^0y^d + z_{1,j}x^1y^{d-1} + \cdots + z_{d,j}x^dy^0,$$

then

$$\boldsymbol{g} \in \mathbb{A}_d \setminus \Phi(\mathrm{BalH}_d) \iff \det W = 0,$$

where W is

W is the $3c \times 3c$ matrix:

$$\begin{bmatrix}
z_{0,1} & 0 & \cdots & 0 & z_{0,2} & 0 & \cdots & 0 & z_{0,3} & 0 & \cdots & 0 \\
z_{1,1} & z_{0,1} & \cdots & 0 & z_{1,2} & z_{0,2} & \cdots & 0 & z_{1,3} & z_{0,3} & \cdots & 0 \\
z_{2,1} & z_{1,1} & \cdots & 0 & z_{2,2} & z_{1,2} & \cdots & 0 & z_{2,3} & z_{1,3} & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
z_{d-1,1} & z_{d-2,1} & \cdots & \vdots & z_{d-1,2} & z_{d-2,2} & \cdots & \vdots & z_{d-1,3} & z_{d-2,3} & \cdots & \vdots \\
z_{d,1} & z_{d-1,1} & \cdots & \vdots & z_{d,2} & z_{d-1,2} & \cdots & \vdots & z_{d,3} & z_{d-1,3} & \cdots & \vdots \\
0 & z_{d,1} & \cdots & \vdots & 0 & z_{d,2} & \cdots & \vdots & 0 & z_{d,3} & \cdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & z_{d,1} & 0 & 0 & \cdots & z_{d,2} & 0 & 0 & \cdots & z_{d,3}
\end{bmatrix}.$$

Each block of columns has c columns for a total of $3c$ columns. There are $(d+1)+(c-1) = d+c = 3c$ rows.

Proof of the answer to Question 1.

Here is the significance of W. If $\boldsymbol{q}$ is a 3×1 matrix with entries from B_{c-1} and $\boldsymbol{b}$ is the column vector of coefficients of $\boldsymbol{q}$, then

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}\boldsymbol{q}=0 \iff W\boldsymbol{b}=0$$

because $\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}\boldsymbol{q}$

$$=\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}\begin{bmatrix} y^{c-1}\dots x^{c-1} & & \\ & y^{c-1}\dots x^{c-1} & \\ & & y^{c-1}\dots x^{c-1} \end{bmatrix}\boldsymbol{b}$$

$$=\begin{bmatrix} y^{d+c-1}\dots x^{d+c-1} \end{bmatrix}W\boldsymbol{b}.$$

So,

$$
\begin{aligned}
\boldsymbol{g} \in \mathbb{A}_d \setminus \Phi(\mathrm{BalH}_d) \quad & \Longleftrightarrow^{*} && \text{there exists a non-zero } 3 \times 1 \text{ matrix } \boldsymbol{q} \\
& && \text{of forms of degree } c-1 \\
& && \text{from } B \text{ with } \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \boldsymbol{q} = 0 \\
& \Longleftrightarrow && \text{there exists a non-zero } 3c \times 1 \text{ matrix } \boldsymbol{b} \\
& && \text{of constants with } W\boldsymbol{b} = 0 \\
& \Longleftrightarrow && \det W = 0. \qquad \square
\end{aligned}
$$

$\Longleftrightarrow$ * THIS is the critical step.

The precise answer to Question 2.

• If $\boldsymbol{g}$ is in $\Phi(\mathrm{BalH}_d)$, then $\det W \neq 0$ and $z_{0,j} \neq 0$ for some $j \in \{1,2,3\}$.

(Otherwise, the ideal (g_1, g_2, g_3) is contained in the ideal (x) and hence has the wrong height.)

• We exhibit σ_1, a section of Φ on the open subset $U_1 = \{\boldsymbol{g} \in \Phi(\mathrm{BalH}_d) \mid z_{0,1} \neq 0\}$.

(The other two members of the open cover of $\Phi(\mathrm{BalH}_d)$ and the other two σ_j are defined similarly.)

Consider the $(3c+1)\times(3c+3)$ matrix A:

$$
\begin{bmatrix}
z_{0,1} & 0 & \cdots & 0 & z_{0,2} & 0 & \cdots & 0 & z_{0,3} & 0 & \cdots & 0 \\
z_{1,1} & z_{0,1} & \cdots & 0 & z_{1,2} & z_{0,2} & \cdots & 0 & z_{1,3} & z_{0,3} & \cdots & 0 \\
z_{2,1} & z_{1,1} & \cdots & 0 & z_{2,2} & z_{1,2} & \cdots & 0 & z_{2,3} & z_{1,3} & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
z_{d-1,1} & z_{d-2,1} & \cdots & \vdots & z_{d-1,2} & z_{d-2,2} & \cdots & \vdots & z_{d-1,3} & z_{d-2,3} & \cdots & \vdots \\
z_{d,1} & z_{d-1,1} & \cdots & \vdots & z_{d,2} & z_{d-1,2} & \cdots & \vdots & z_{d,3} & z_{d-1,3} & \cdots & \vdots \\
0 & z_{d,1} & \cdots & \vdots & 0 & z_{d,2} & \cdots & \vdots & 0 & z_{d,3} & \cdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & z_{d,1} & 0 & 0 & \cdots & z_{d,2} & 0 & 0 & \cdots & z_{d,3}
\end{bmatrix}.
$$

Each block of columns has $c+1$ columns for a total of $3c+3$ columns.

There are $(d+1)+(c) = 3c+1$ rows.

Here is the significance of A. If $\boldsymbol{q} \in \mathrm{Mat}_3(B_c)$ and $\boldsymbol{b}$ is the column vector of coefficients of $\boldsymbol{q}$, then

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix} \boldsymbol{q} = 0 \iff A\boldsymbol{b} = 0.$$

To describe a Hilbert-Burch matrix for $\boldsymbol{g}$, we need only to produce two linearly independent relations $\boldsymbol{q} \in \mathrm{Mat}_3(B_c)$.* We carefully choose two Eagon-Northcott relations on A.

*THIS is the critical observation.

Cross out column $c+2$ of A. (This is the FIRST column of the SECOND block of columns. The maximal minors of the resulting $(3c+1)\times(3c+2)$ matrix become the relation

$$\boldsymbol{q}_2 = \begin{bmatrix} *y^c + *xy^{c-1} + \cdots + *x^c \\ 0y^c + *xy^{c-1} + \cdots + *x^c \\ -\boldsymbol{\Delta} y^c + *xy^{c-1} + \cdots + *x^c \end{bmatrix}$$

on $[g_1, g_2, g_3]$, where $\boldsymbol{\Delta}$ is the determinant of A with columns $c+2$ and $2c+3$ removed. Cross out column $2c+3$ of A. This is the FIRST column of the THIRD block of columns.) The resulting relation is

$$\boldsymbol{q}_3 = \begin{bmatrix} *y^c + *xy^{c-1} + \cdots + *x^c \\ (-1)^{c+1}\boldsymbol{\Delta} y^c + *xy^{c-1} + \cdots + *x^c \\ 0y^c + *xy^{c-1} + \cdots + *x^c \end{bmatrix}.$$

We have $\boldsymbol{\Delta} = z_{0,1} \det W \neq 0$. We see that $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are $\neq 0$ and lin. indept..

The precise answer to Question 2.

Recall the open subset $U_1 = \{\boldsymbol{g} \in \Phi(\mathrm{BalH}_d) \mid z_{0,1} \neq 0\}$ of $\Phi(\mathrm{BalH}_d)$.

Theorem. If $\sigma_1 : U_1 \to \mathbb{H}_d$ is defined by

$$\sigma_1(\boldsymbol{g}) = \left[\frac{1}{z_{0,1}(\det W)^2}\boldsymbol{q}_2 \quad \boldsymbol{q}_3 \right],$$

then $\Phi \circ \sigma_1$ is the identity map on U_1.

Extensions:

• Question. What happens when one considers

$$\Phi : \mathrm{Mat}_{n+1,n}(B_c) \to \mathrm{Mat}_{1,n+1}(B_{nc})$$

instead of

$$\Phi : \mathrm{Mat}_{3,2}(B_c) \to \mathrm{Mat}_{1,3}(B_{2c})?$$

Answer. (Brett Barwick) One gets “the same” answer.

• Question. What happens when d is odd? Say $d = 2c+1$. Can one find a local section of the map

$$\Phi : \left\{ M = (m_{i,j}) \in \mathrm{Mat}_{3,2} \;\middle|\; \begin{array}{l} m_{i,1} \in B_c, \\ m_{i,2} \in B_{c+1}, \\ \mathrm{ht}(\mathrm{I}_2(\mathrm{M})) = 2 \end{array} \right\} \to \mathrm{Im} \subseteq \mathrm{Mat}_{1,3}(B_{2c+1})?$$

• Question 2, repeated. Suppose $d = 2c+1$. Can one find a local section of the map

$$\Phi : \left\{ M = (m_{i,j}) \in \mathrm{Mat}_{3,2} \;\middle|\; \begin{array}{l} m_{i,1} \in B_c, \\ m_{i,2} \in B_{c+1}, \\ \mathrm{ht}(\mathrm{I}_2(\mathrm{M})) = 2 \end{array} \right\} \to \mathrm{Im} \subseteq \mathrm{Mat}_{1,3}(B_{2c+1})?$$

Work in progress. (Brett Barwick) One can again describe explicitly the equations which define the complement of Im. There is promising evidence that one can again define a local section. One must use a much larger open cover of Im.

Other Interpretations:

One can build $G_j = \sum_{i=0}^{d} z_{i,j} x^i y^{d-i}$, for $1 \le j \le 3$, in $S = \mathbb{Z}[\{z_{i,j}\}][x,y]$. One can also build W, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$. Let $w = \det W$.

Theorem. The following three statements hold.

(1) $(G_1, G_2, G_3)S_w$ is a perfect height two ideal of S_w.

(2) If $\mathbb{F}$ is the complex

$$0 \to S(-3c, -3c-3) \xrightarrow{\boldsymbol{d}_3} S(-3c, -3c-2)^2 \xrightarrow{\boldsymbol{d}_2} S(-2c, -1)^3 \xrightarrow{\quad \boldsymbol{d}_1 \quad} S,$$

with

$$\boldsymbol{d}_3 = \begin{bmatrix} z_{0,1} \\ z_{0,2} \\ z_{0,3} \end{bmatrix}, \quad \boldsymbol{d}_2 = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \end{bmatrix}, \quad \boldsymbol{d}_1 = \begin{bmatrix} G_1, G_2, G_3 \end{bmatrix},$$

then $\mathbb{F}_w$ is a free resolution of $S_w/(G_1, G_2, G_3)S_w$.

(3) We split $S_w \xrightarrow{\boldsymbol{d}_3} S_w$ from $\mathbb{F}_w$ to produce a Universal Projective Resolution for the graded Betti numbers:

$$0 \to B(-3c)^2 \to B(-2c)^3 \to B.$$

(The base ring for this UPR is S_w.)

Some consequences of

"(1) $(G_1, G_2, G_3)S_w$ is a perfect height two ideal of S_w."

are

(a) $S_w/(G_1, G_2, G_3)S_w$ is a Cohen-Macaulay ring,

(b) $\text{grade}_{S_w}(G_1, G_2, G_3) = \text{pd}_{S_w} S_w/(G_1, G_2, G_3)S_w = 2$,

(c) (The Persistence of Perfection Principal) if N is a noetherian S_w-algebra and $(G_1, G_2, G_3)N$ is a proper ideal of N of grade at least 2, then $(G_1, G_2, G_3)N$ is a perfect ideal of grade equal to 2 and $\mathbb{F}_w \otimes_{S_w} N$ is a resolution of $N/(G_1, G_2, G_3)N$.