Analysis II Review (Test 2)

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April 13, 2009

1 DEFINITIONS

1.1 Integral of Nonnegative Meas Function

Let E be a measurable set, and let $f : E \to [0, \infty]$ be measurable. Then we define

$$\int_E f\,dx = \sup\left\{\int_E \varphi\,dx : \varphi \text{ simple, } 0 \le \varphi \le f\right\}$$

If $E = \mathbb{R}^d$, then we write $\int f\,dx$.

1.2 L^p Space

For $0 < p < \infty$, we define

$$L^p = \{f : f \text{ is measurable and } |f|^p \text{ is integrable}\}$$

For $p = \infty$, we have

$$L^\infty = \{f : f \text{ measurable and } |f(x)| \le M \text{ a.e.}\}$$

In this case, we say f is **essentially bounded**.

1.3 L^p Norm

For $1 \le p < \infty$, we define

$$\|f\|_p = \left(\int |f|^p\,dx\right)^{1/p}$$

For $p = \infty$, we define

$$\|f\|_\infty = \inf\{M \ge 0 : |f(x)| \le M \text{ for a.e. } x\}$$

1.4 Important Properties of L^p Spaces

a) If $f \in L^\infty$, then $|f(x)| \le \|f\|_\infty$ a.e.

b) L^∞ is a Banach space (normed complete vector space)

c) L^p is a Banach space for $1 \le p < \infty$, but this takes more work (Riesz-Fisher Theorem).

1.5 Hardy-Littlewood Maximal Function

If $f \in L^1(\mathbb{R}^d)$, we define its **maximal function** f^* by

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)|\,dy, \quad x \in \mathbb{R}^d$$

1.6 Lebesgue Set

Recall that f is **locally integrable** if for every ball B the function $f(x)\chi_B(x)$ is integrable. If $f \in L^1_{\text{loc}}$, then we define the **Lebesgue set** of f as

$$\left\{ x : f(x) \text{ is finite, } \lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B |f(y) - f(x)|\, dy = 0 \right\}$$

Note that if x is in the Lebesgue set of f, then

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B f(y)\, dy = f(x)$$

1.7 Function of Bounded Variation

Let $F : [a,b] \to \mathbb{R}$. We say F is of **bounded variation** if

$$\sup_{\mathcal{P}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \infty$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[a,b]$.

1.8 Dini Derivates

We define the **Dini Derivates** of a function F by

$$D^+F(x) = \limsup_{h\downarrow 0} \frac{F(x+h) - F(x)}{h}$$

$$D_+F(x) = \liminf_{h\downarrow 0} \frac{F(x+h) - F(x)}{h}$$

$$D^-F(x) = \limsup_{h\downarrow 0} \frac{F(x) - F(x-h)}{h}$$

$$D_-F(x) = \liminf_{h\downarrow 0} \frac{F(x) - F(x-h)}{h}$$

If $D^+F(x) = D_+F(x) = D^-F(x) = D_-F(x)$, the we say $F'(x)$ exists since

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \qquad \text{exists.}$$

If, in addition, $F'(x)$ is finite, then we say F is differentiable at x.

2 THEROEMS

2.1 Fatou's Lemma

If $0 \le f_n$ measurable, then $\int \underline{\lim} f_n dx \le \underline{\lim} \int f_n dx$. In particular, if $f_n \to f(x), \forall x$, then $\int f dx \le \underline{\lim} \int f_n dx$.

2.2 Monotone Convergence Theorem

Let f_n measurable, $\forall n$ and $0 \le f_1 \le f_2 \le \cdots \uparrow$ and assume $f(x) = \lim_{n\to\infty} f_n(x), \forall x$. Then, $\int f dx = \lim_{n\to\infty} \int f_n(x)$.

2.3 Dominated Convergence Theorem

If f_n, g integrable, $f_n(x) \to f(x)$ a.e., and $|f_n| \le g$ a.e. Then, f is integrable and

$$\int f dx = \lim_{n\to\infty} \int f_n(x).$$

2.4 Riesz-Fisher Theorem., i.e., the completeness of L^p

L^p is a Banach space for $1 \le p < \infty$.

2.5 Hölder's Inequality

Let $1 \le p \le \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $f \in L^p, g \in L^q$ implies $fg \in L^1$, and

$$\int |fg| dx \le \|f\|_p \|g\|_q.$$

Moreover, for $1 < p < \infty$, equality holds $\Leftrightarrow$ $\{|f|^p, |g|^q\}$ are linearly dependent. i.e. $|f|^p = c|g|^q$ for some constant $c \neq 0$.

2.6 Minkowski's Inequality

Let $1 \le p < \infty$ and $f, g \in L^p$. Then $f + g \in L^p$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

2.7 Fubini's Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable, where $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $d = d_1 + d_2$. Then
(i) for a.e. $y \in \mathbb{R}^{d_2}$, the slice f^y is integrable.
for a.e. $x \in \mathbb{R}^{d_1}$, the slice f_x is integrable.
(ii) the function $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$ is integrable.
the function $x \mapsto \int_{\mathbb{R}^{d_2}} f_x(y) dy$ is integrable.
(iii)

$$\int_{\mathbb{R}^{d_2}} \Big[\int_{\mathbb{R}^{d_1}} f^y(x) dx \Big] dy = \int_{\mathbb{R}^d} f.$$

$$\int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} f_x(y) dy \Big] dx = \int_{\mathbb{R}^d} f.$$

In particular,

$$\int_{\mathbb{R}^{d_2}} \Big[\int_{\mathbb{R}^{d_1}} f(x,y) dx \Big] dy = \int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} f(x,y) dy \Big] dx$$

2.8 Tonelli's Theorems

Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable and $f \ge 0$ a.e., where $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $d = d_1 + d_2$. Then
(i) for a.e. $y \in \mathbb{R}^{d_2}$, the slice f^y is measurable.
for a.e. $x \in \mathbb{R}^{d_1}$, the slice f_x is measurable.
(ii) the function $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$ is measurable.
the function $x \mapsto \int_{\mathbb{R}^{d_2}} f_x(y) dy$ is measurable.
(iii)

$$\int_{\mathbb{R}^{d_2}} \Big[\int_{\mathbb{R}^{d_1}} f^y(x) dx \Big] dy = \int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} f_x(y) dy \Big] dx = \int_{\mathbb{R}^d} f.$$

In particular, if one of the repeated integrals is finite, then f is integrable.

2.9 Weak L^1 estimate for f^*

Suppose f is integrable on R^d. Then
(i) f^* is measurable
(ii) $f^*(x) < \infty$ for a.e. x
(iii) f^* satisfies

$$m(x \in \mathbb{R}^d : f^*(x) > \alpha) \leq \frac{3^d}{\alpha}||f||_1 \text{ for all } \alpha > 0.$$

2.10 Lebesgue's Differentiation Theorem

If f is integrable on $\mathbb{R}^d$, then

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B f(y)dy = f(x) \quad for\ a.e.\ x$$

2.11 Decomposition of Functions of Bdd Variation

Let F be a continuous function of bounded variation. Then $F = F_1 - F_2$, where both F_1 and F_2 are monotonic and continuous. Namely $F_1 = P_a^x(F)$ and $F_2 = N_a^x(F) + F(a)$.

2.12 Lebesque's Thm of Differentiation of Increasing Functions

Let $F : [a, b] \to \mathbb{R}$ be an increasing function. Then F is differentiable a.e., F' is measurable, non-negative, and $\int_a^b F'dx \leq F(b) - F(a)$.

3 EXAMPLES

3.1 Illustrative Examples

3.1.1 Fatou's Lemma (2/24)

Let $f_n \geq 0$, $\int f_n dx \leq 1\ \forall n$, and $f_n(x) \to f(x)\ \forall x$. Then $\int f dx \leq 1$.

3.1.2 Corollary to MCT (2/24)

Prove the following series is finite almost everywhere.

$$\sum_{n=1}^{\infty} \frac{1}{2^n|x - r_n|^{1/2}}$$

3.1.3 Lebesgue and Riemann Integrable (2/26-3/3)

Let $\mathcal{C}$ be the Cantor set. Then $\chi_\mathcal{C}$ is continuous on $[0, 1] - \mathcal{C}$ but is discontinuous on $\mathcal{C}$. So $\chi_\mathcal{C}$ is continuous a.e. Hence $\chi_\mathcal{C}$ is Riemann integrable. And $\chi_\mathcal{C}$ is Lebesque integrable and $\int \chi_\mathcal{C} dx = m(\mathcal{C}) = 0$.

3.1.4 Tonelli's Corollary (3/24)

Let $E = \{(x, y) \in \mathbb{R}^2 : y = ax + b \text{ for some } a, b \in \mathbb{R}\}$. Then $m(E) = 0$ as $m(E^y) = 0 \forall y$.

3.1.5 Measurability of x and y slices (3/24)

- If $E \subset \mathbb{R}^d$ Borel, then E^y is measurable for all y
- Let $f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$. If $f^{-1}([a, \infty])$ is Borel for all a, then $f^y\ \forall y$, $f_x\ \forall x$ are measurable.

3.1.6 Shrinks Regularly to x (3/31)

- The set of all closed cubes shrinks regularly to x.
- $\{[x, x+h] : h > 0\}$ shrinks regularly to x.

3.1.7 Functions of Bounded Variation (4/2)

- If F is increasing, then F is of bounded variation and $T_a^b(F) = F(b) - F(a)$.
- If F is decreasing, then F is of bounded variation and $T_a^b(F) = F(a) - F(b)$.
- Apply the first 2 to functions that increase and decrease on intervals.

3.1.8 $\int_a^b F' dx$ strictly less than $F(b) - F(a)$

Let f be the Cantor Lebesgue function. Then $0 = \int_0^1 f' dx < f(1) - f(0) = 1$.

3.2 Counter Examples

3.2.1 (2/26) $lim_{n\to\infty} \int f_n dx = \int f dx$ does not imply $lim_{n\to\infty} \int |f_n(x) - f(x)| dx \to 0$

- $f_n = \chi_{[n,n+1/2]} - \chi_{[n+1/2,n+1]}$. Then $f_n \to 0 \forall x$, $\int f_n dx = 0\ \forall n$. BUT $\int |f_n - 0| dx = 1\ \forall n$.
- $f_n = n[\chi_{[1/(n+1),a_n]} - \chi_{[a_n,1/n]}]$ where $a_n = 1/2(1/n + \frac{1}{n+1})$. Then $f_n \to 0 \forall x$, $\int f_n dx = 0\ \forall n$. BUT $\int |f_n - 0| dx = 1\ \forall n$.

3.2.2 Lebesgue Integrable but not Riemann Integrable (2/26-3/3)

- $f = \chi_{\mathbb{Q}\cap[0,1]}$. Then f is not Riemann integrable as f is discontinuous at every $x \in [0,1]$. BUT f is Lebesgue integrable and $\int f dx = 0$.
- Let E be a Cantor-like set, and $m(E) > 0$. Then χ_E is continuous on $[0,1] - E$ but is discontinuous on E. So χ_E is not continuous a.e. Hence χ_E is not Riemann integrable. BUT χ_E is Lebesque integrable and $\int \chi_E dx = m(E) = 0$.

3.2.3 Riemann Integrable but not Lebesgue Integrable (3/3)

- $f = \frac{\sin(x)}{x}$. $\int_0^\infty f dx$ exists as an improper Riemann Integral, BUT not as a Lesbesgue integral.
- $$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \chi_{[n,n+1]}(x)$$

  Then $\int_0^\infty f dx = \sum_{n=1}^\infty \frac{(-1)^n}{n} < \infty$. BUT $\int_{[0,\infty)} |f| dx = \sum_{n=1}^\infty \frac{(1)^n}{n} = \infty$ by the MCT.

3.2.4 L^∞ is not seperable (3/17)

Consider $\{\chi_{[0,x]} : x \geq 0\}$. If $x \neq y$ then $||\chi_{[0,x]} - \chi[0,y]||_\infty = 1$. If A dense in L^∞, then there exists $\epsilon \leq 1/2$ and a_x s.t. $||a_x - \chi_{[0,x]}||_\infty < 1/2$. Now $a_x \neq a_y$ if $x \neq y$. Therefore A is uncountable.

3.2.5 E^y measurable $\forall y \in \mathbb{R}^{d_2}$ does not imply E is measurable (3/24)

- Let $E = [0,1] \times \mathcal{N} \subset \mathbb{R}^2$. Then E^y is measurable for all $y \in \mathcal{N}$, BUT E_x is not measurable for all $x \in [0,1]$.
- Assuming Continuim Hypothesis, the $\exists E \subset [0,1] \times [0,1]$ s.t. for a.e. y, E^y is measurable and for a.e. x, E_x is measurable, BUT E is not measurable.

3.2.6 $F(x) \neq F(a) + \int_a^x F'(y)dy$ (3/26)

Let F be the Cantor-Lebesque function on $[0,1]$. Then $F(0)=0,\ F(1)=1,\ F'(x)=0$ a.e. So $F(x) \neq F(0)+ \in_0^x F'(y)dy = 0$.

3.2.7 Function not of Bounded Variation (4/2)

Function that alternates between the lines $y=x$ and $y=-x$ at values $1/n$. $T_0^1(F) \geq \sum 1/n = \infty$. Note F' is piece-wise constant, but not integrable.

4 Key Results from Homework

- Integrability does not imply convergence to 0 (6.6)
  - Define f to be a sequence of "spikes" having small area (say, a triangle at each n having area $\frac{1}{n^2}$). This function is continuous and integrable, but does not converge to 0.
  - If, in addition, we assume that f is uniformly continuous, then we have that $f \to 0$.
- If $f : \mathbb{R}^d \to \mathbb{R}$ is integrable, real-valued, and $\int_E f(x)dx \geq 0$ for all measurable E, then $f(x) \geq 0$ a.e. (6.11)
- $\liminf(a_n + b_n) = a + \liminf b_n$ (6.ex1a)
- If f, f_n integrable with $f_n(x) \to f(x)$ a.e. and $\int |f_n|dx \to \int |f|dx$, then $\int |f_n - f|dx \to 0$ (6.ex1b)
  - This holds more generally: Let $f_n, f \in L^p$ with $1 \leq p < \infty$ and $f_n(x) \to f(x)$ a.e. Then $f_n \to f$ in L^p if and only if $||f_n||_p \to ||f||_p$. (8.ex2)
- For E of finite measure, $L^\infty(E) \subset L^r(E) \subset L^p(E) \subset L^1(E)$. (7.1)
  - These inclusions are all strict (play around with things like $f(x) = \frac{1}{x^{\frac{1}{r}}}$ so that applying the power r gives you a nonintegrable function).
  - This does not hold for E of infinite measure.