Analysis II Final Review

Kenny Brown, Jing Liu, Kamala Diefenthaler

April 27, 2009

1 DEFINITIONS

1.1 Exterior (Outer) Measure

Let E be any subset of $\mathbb{R}^d$. Then the **exterior measure** of E is the inf over all countable covering of E by closed cubes, rectangles, or balls.

1.2 (Lebesgue) Measurable Set

E a subset of $\mathbb{R}^d$ is **Lebesgue measurable** if for any $\epsilon > 0$ there exists an open set O with $E \subset O$ and $m^*(O - E) \leq \epsilon$.

1.3 Measurable Function

A function $f : E \to \mathbb{R}^d$ is **measurable** if for all $a \in \mathbb{R}$, the set $\{f < a\}$ is measurable.
$\{f < a\}$ measurable $\Leftrightarrow$ $\{f \leq a\}$ measurable $\Leftrightarrow$ $\{f > a\}$ measurable $\Leftrightarrow$ $\{f \geq a\}$ measurable

1.4 Convergence in Measure

Let $f_n, f : E \to \mathbb{R}$ measurable. Then $f_n \to f$ in measure if $\forall \epsilon > 0$ $\exists N$ s.t. $m(\{x \in E : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$ $\forall n \geq N$.

1.5 Function of Bounded Variation

Let $F : [a, b] \to \mathbb{R}$. We say F is of **bounded variation** if

$$\sup_{\mathcal{P}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \infty$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[a, b]$.

1.6 Absolute Continuity

$F : [a, b] \to \mathbb{R}$ is **absolutely continuous** if $\forall\ \epsilon > 0 \quad \exists\ \delta > 0$ s.t. if $\{(a_i, b_i)\}_{i=1}^{n}$ is a collection of disjoint open intervals in $[a, b]$ with $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon$.

2 THEROEMS

2.1 Properties of Outer Measure

- (Monotonicity) If $E_1 \subset E_2$, then $m^*(E_1) \leq m^*(E_2)$.
- (Countable Sub-additivity) If $E = \cup_{j=1}^{\infty} E_j$, then $m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j)$.
- For any set E and $\epsilon > 0$, there exists an open set $O \supset E$ s.t. $m^*(O) \leq m^*(E) + \epsilon$, as $m^*(E) = \inf m^*(O)$ for all open sets O containing E.

2.2 Properties of Measurable Sets

- Every open set in $\mathbb{R}^d$ is measurable.
- If $m^*(E) = 0$, then E is measurable.
- A countable union of measurable sets is measurable.
- Every closed set in $\mathbb{R}^d$ is measurable.
- The compliment of a measurable set is measurable.
- A countable intersection of measurable sets is measurable.

2.3 G_δ and F_σ approximation of measurable sets

A subset E of $\mathbb{R}^d$ is measurable

- if and only if E differs from a G_δ set by a set of measure 0.
  G_δ sets are countable intersections of open sets.
- if and only if E differs from a F_σ set by a set of measure 0.
  F_σ sets are countable unions of closed sets.

2.4 Properties of Measurable Functions

- The finite value function f is measurable $\iff$
  $f^{-1}(O)$ is measurable for every open set O and $\iff$
  $f^{-1}(F)$ is measurable for every closed set F.
- If f is continuous on $\mathbb{R}^d$, then f is measurable.
  If f is measurable and finite-valued, and ϕ is continuous, then $\phi \circ f$ is measurable.
- Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions. Then sup, inf, limsup, and liminf of $f_n(x)$ are measurable.
- Suppose $\{f_n\}_{n=1}^\infty$ is a collection of measurable functions, and $\lim_{n\to\infty} f_n(x) = f(x)$. Then f is measurable.
- If f and g are measurable, then
  (i) f^k where $k = 1, 2, 3, ...$ are measurable.
  (ii) For f and g finite-valued, $f + g$ and fg are measuable.
- If f is measurable and $f(x) = g(x)$ for *a.e.* x, then g is measurable.

2.5 Approximation by Simple Functions

Suppose $f : E \to [0, \infty]$ is measurable. Then there exists a sequence of simple functions $\{\phi_k\}_{k=1}^\infty$ that satisfies

$$|\phi_k(x)| \leq |\phi_{k+1}(x)|, \text{ and } \lim_{k\to\infty} \phi_k(x) = f(x), \text{ for all } x \in E$$

Moreover if f is bounded, then $\{\phi_k\}$ converges uniformly to f.

2.6 Egorov's Theorem

Suppose $\{f_k\}_{k=1}^\infty$, $f_k : E \to \mathbb{R}^d$ measurable with $m(E) < \infty$, and assume that $f_k \to f$ *a.e.* on E.
Given $\epsilon > 0$, we can find a (closed) set $A_\epsilon \subset E$ s.t. $m(E - A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ.
Think of A_ϵ as the "good set" and $E - A_\epsilon$ as the "bad set".

2.7 Relation between $a.e.$ and convergence in measure

Let $f_n, f : E \to \mathbb{R}^d$ measurable s.t. $m(\{x \in E : |f_n(x) - f(x)| \geq 1/2^n\}) < 1/2^n$. Then $f_n(x) \to f(x)$ $a.e.$

If (f_n) converges in measure to f on E, then $\exists (f_{n_k}) \subset (f_n)$ s.t. $f_{n_k}(x) \to f(x)$ $a.e.$.

If $m(E), \infty$ and $f_n \to f$ $a.e.$ on E, then $f_n \to f$ in measure.

2.8 Fatou's Lemma

If $0 \leq f_n$ measurable, then $\int \underline{\lim} f_n dx \leq \underline{\lim} \int f_n dx$. In particular, if $f_n \to f(x), \forall x$, then $\int f dx \leq \underline{\lim} \int f_n dx$.

2.9 Monotone Convergence Theorem

Let f_n measurable, $\forall n$ and $0 \leq f_1 \leq f_2 \leq \cdots \uparrow$ and assume $f(x) = \lim\limits_{n\to\infty} f_n(x), \forall x$. Then, $\int f dx = \lim\limits_{n\to\infty} \int f_n(x)$.

2.10 Dominated Convergence Theorem

If f_n, g integrable, $f_n(x) \to f(x)$ a.e., and $|f_n| \leq g$ a.e. Then, f is integrable and

$$\int f dx = \lim_{n\to\infty} \int f_n(x).$$

2.11 Riesz-Fisher Theorem., i.e., the completeness of L^p

L^p is a Banach space for $1 \leq p < \infty$.

2.12 Hölder's Inequality

Let $1 \leq p \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $f \in L^p, g \in L^q$ implies $fg \in L^1$, and

$$\int |fg| dx \leq \|f\|_p \|g\|_q.$$

Moreover, for $1 < p < \infty$, equality holds $\Leftrightarrow \{|f|^p, |g|^q\}$ are linearly dependent. i.e. $|f|^p = c|g|^q$ for some constant $c \neq 0$.

2.13 Minkowski's Inequality

Let $1 \leq p < \infty$ and $f, g \in L^p$. Then $f + g \in L^p$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

2.14 Fubini's Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable, where $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $d = d_1 + d_2$. Then
(i) for a.e. $y \in \mathbb{R}^{d_2}$, the slice f^y is integrable.
for a.e. $x \in \mathbb{R}^{d_1}$, the slice f_x is integrable.
(ii) the function $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$ is integrable.
the function $x \mapsto \int_{\mathbb{R}^{d_2}} f_x(y) dy$ is integrable.
(iii)

$$\int_{\mathbb{R}^{d_2}} \Big[\int_{\mathbb{R}^{d_1}} f^y(x) dx \Big] dy = \int_{\mathbb{R}^d} f.$$

$$\int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} f_x(y) dy \Big] dx = \int_{\mathbb{R}^d} f.$$

In particular,

$$\int_{\mathbb{R}^{d_2}} \Big[\int_{\mathbb{R}^{d_1}} f(x, y) dx \Big] dy = \int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} f(x, y) dy \Big] dx$$

2.15 Tonelli's Theorems

Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable and $f \geq 0$ a.e., where $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $d = d_1 + d_2$. Then
(i) for a.e. $y \in \mathbb{R}^{d_2}$, the slice f^y is measurable.
for a.e. $x \in \mathbb{R}^{d_1}$, the slice f_x is measurable.
(ii) the function $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x)dx$ is measurable.
the function $x \mapsto \int_{\mathbb{R}^{d_2}} f_x(y)dy$ is measurable.
(iii)

$$\int_{\mathbb{R}^{d_2}} \Big[\int_{\mathbb{R}^{d_1}} f^y(x)dx\Big]dy = \int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} f_x(y)dy\Big]dx = \int_{\mathbb{R}^d} f.$$

In particular, if one of the repeated integrals is finite, then f is integrable.

2.16 Weak L^1 estimate for f^*

Suppose f is integrable on R^d. Then
(i) f^* is measurable
(ii) $f^*(x) < \infty$ for a.e. x
(iii) f^* satisfies

$$m(x \in \mathbb{R}^d : f^*(x) > \alpha) \leq \tfrac{3^d}{\alpha}||f||_1 \text{ for all } \alpha > 0.$$

2.17 Lebesgue's Differentiation Theorem

If f is integrable on $\mathbb{R}^d$, then

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)} \int_B f(y)dy = f(x) \quad for\ a.e.\ x$$

2.18 Decomposition of Functions of Bdd Variation in terms of increasing functions

Let F be a continuous function of bounded variation. Then $F = F_1 - F_2$, where both F_1 and F_2 are increasing and continuous. Namely $F_1 = P_a^x(F)$ and $F_2 = N_a^x(F) + F(a)$.

2.19 Lebesque's Thm of Differentiation of Increasing Functions

Let $F : [a, b] \to \mathbb{R}$ be an increasing function. Then F is differentiable a.e., F' is measurable, non-negative, and $\int_a^b F'dx \leq F(b) - F(a)$.

2.20 2^{nd} Fundamental Thm of Calculus for the Lebesgue Integral***

Let $F : [a, b] \to \mathbb{R}$. F is absolutely continuous $\iff$ $F'(x)$ exists a.e., $F'(x) \in L^1([a, b])$, and
$F(x) = F(a) + \int_a^x F'(y)dy$.