Analysis Study Guide

Kamala Hunt Diefenthaler

December 9, 2008

1 DEFINITIONS

1.1 (Sequentially) Compact

A set A is sequentially compact $\Leftrightarrow$ every sequence in A has a convergent subsequence with limit in A.

1.2 Open Covering

A collection $\mathcal{C}$ of open subsets of X is called an open covering of $A \subset X$, if $A \subset \bigcup\{C : C \in \mathcal{C}\}$.

1.3 (Topologically) Compact

A is topologically compact, if every open covering of A has a fintite subcovering.

1.4 Uniform Continuity

Let $f : X \to Y$ be a function. We say that f is uniformly continuous if, for every $\epsilon > 0$, there exists a $\delta > 0$ s.t. $d(f(x), f(x_0)) < \epsilon$ whenever $x, x_0 \in X$ are such that $d(x, x_0) < \delta$.
(Sequential Version useful for counter examples) f is uniformly continuous if for all $(x_n), (y_n)$ $d(x_n, y_n) \to 0 \Rightarrow d(f(x_n), f(y_n)) \to 0$

1.5 Uniform Convergence

Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from (X, d_x) to (Y, d_y), and let $f : X \to Y$.
(f_n) converges uniformly to f on X if $\forall\epsilon > 0, \exists N > 0$ s.t. $d_y(f_n(x), f(x)) < \epsilon$ for every $n > N$ and $x \in X$.

1.6 Complex Differentiability

Let $E \subseteq \mathbb{C}$, and $z_o \in E$ be a limit point. Then $f : E \to \mathbb{C}$ is differentiable at z_o if $\lim_{z\to z_o} \frac{f(z)-f(z_o)}{z-z_o}$ exists.

1.7 Cauchy Riemann Equations

Let $f(z) = u(x, y) + i * v(x, y)$. If $f(z)$ is differentiable the the Cauchy Riemann equations hold. Thus $u_x = v_y$ and $u_y = -v_x$.

1.8 Starlike Region

$S \subseteq \mathbb{C}$ is starlike if there exists an $a \in S$ such that $[a, z] \subset S$ $\forall z \in S$. Furthermore, a is called the starcenter of S.

1.9 Isolated Singularity

G open, $z_0 \in G$, f analytic on $G\backslash\{z_0\}$. Then z_0 is called an isolated singularity of f.

1.10 Removable Singularity

An isolated singularity z_0 of f is called a removable singularity of f if we can define $f(z_0)$ in such a way that f is analytic on all of G.

1.11 Pole

An isolated singularity z_0 of f is a pole of f if $\lim_{z \to z_0} |f(z)| = \infty$.

1.12 Essential Singularity

An isolated singularity z_0 of f is called an essential singularity of f if z_0 is not removable or a pole.

2 Know the CONTENT

2.1 Statements

- Unions of open sets are open. Finite intersections of open sets are open.
- Intersections of closed sets are closed. Finite unions of closed sets are closed.
- An open subset of $\mathbb{R}^n$ is connected $\Leftrightarrow$ it is pathwise connected.
- Compact Sets are closed and bounded.
  Closed subsets of compact sets are compact.
- Sequentially compact $\Leftrightarrow$ Topologically Compact $\Leftrightarrow$ Complete and Totally Bounded
- If f is continuous, then $f(O)$ open $\Rightarrow O$ open.
- A continuous image of a compact set is compact.
- A continuous function on a compact set has a max and a min.
- A continuous function on a compact set is uniformly continuous.
- Uniform limit of a sequence of continuous functions is continuous.
- Power series with positive radius of convergence are (1) analytic and (2) its derivative is the sum of term wise differenitated series.

2.2 Theorems

2.2.1 Heine-Borel Thm

A subset of $\mathbb{R}^n$ is compact $\Leftrightarrow$ it is closed and bounded.

2.2.2 Cauchy's Theorem for starlike sets

Let $G \subseteq \mathbb{C}$ be open and starlike (connected). Let $p \in G$, f continuous on G and f analytic on $G \diagdown \{p\}$. Then there exists $F : G \to \mathbb{C}$ s.t. $F'(z) = f(z) \forall z \in G$.
Hence $\int_\gamma f(z)dz = 0$ for all γ closed and peicewise smooth.

2.2.3 Cauchy's Integral Formula for starlike sets

Let $G \subseteq \mathbb{C}$ be open and starlike, γ be a piecewise smooth closed curve in G. Let f be analytic on G, and $z \in G \diagdown \gamma^*$. Then

$$f(z) * ind_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw$$

2.2.4 Power series representation of analytic functions

Any analytic function has a power series.
Let $G \subseteq \mathbb{C}$ open, f analytic on G. $\forall z_0 \in G$ and $\forall R > 0$ such that $\mathrm{B}(z_0, R) \subseteq G$, $\exists a_n \in \mathbb{C}$ unique s.t. $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ on $\mathrm{B}(z_0, R)$.
Furthermore, $a_n = \frac{f^{(n)}(z_0)}{n!}$.

2.2.5 Morera's Thm

Let $G \subseteq \mathbb{C}$ open, $f : G \to \mathbb{C}$ continuous. If $\int_{\partial\Delta} f(z)dz = 0\ \forall\Delta \subset G$, then f is analytic. (Cauchy Converse)

2.2.6 Liouville's Thm

Let f be an entire function on $\mathbb{C}$. If $|f|$ is bounded on $\mathbb{C}$, then f is constant.
(i.e. A bounded entire function is constant)

2.2.7 Thm on isolated zeros

Let G open and connected and f analytic on G. Then either every zero of f is isolated or $f \equiv 0$.

2.2.8 Uniqueness Thm

Let G open and connected. Let $S \subset G$ with limit point $z_0 \in G$. If f, g analytic on G and $f(z) = g(z)\ \forall z \in S$, then $f \equiv g$.

2.2.9 Thm on removable singularity

Let $G \subset \mathbb{C}$ open, $z_0 \in G$ isolated singularity of f. If f is bounded on $\mathrm{B}(z_0, r) \setminus \{z_0\}$ for some $r > 0$, then z_0 is a removable singularity.

2.2.10 Characterizations of a pole

Let $G \subset \mathbb{C}$ open, $z_0 \in G$, f analytic on $G\backslash\{z_0\}$. Then the following are equivalent

- f has a pole at z_0
- $\exists m \geq 1$ and $\exists g$ analytic on G, $g(z_0) \neq 0$ s.t.
$$f(z) = \frac{g(z)}{(z - z_0)^m} \forall z \in G\backslash\{z_0\}$$
- $\exists m \geq 1$ and $\exists c_{-1}, ..., c_{-m} \in \mathbb{C}, c_{-m} \neq 0$ s.t. $f(z) - \sum_{k=1}^{m} \frac{c_{-k}}{(z-z_0)^k}$ has a removable singularity at z_0.

2.2.11 Casorati-Weierstrass Thm

Let $G \subset \mathbb{C}$ open, $z_0 \in G$, f analytic on $G\backslash\{z_0\}$. Then the following are equivalent

- z_0 is an essential singularity
- $\forall A \in \mathbb{C}\ \exists z_n \to z_0$ s.t. $\lim_{n\to\infty} f(z_n) = A$
- $\exists z_n \to z_0$ and $\exists w_n \to z_0$ s.t. $\lim_{n\to\infty} f(z_n)$ exists, $\lim_{n\to\infty} f(w_n)$ exists, and $\lim_{n\to\infty} f(z_n) \neq \lim_{n\to\infty} f(w_n)$.

2.2.12 Maximum Modulus Theorem

- Let G be an open and connected set, and $f: G \to \mathbb{C}$ analytic. Assume $\exists z_o \in G$ s.t. $|f(z)| \leq |f(z_o)| \forall z \in G$. Then f is constant.
- If $|f|$ has a maximum on G, then f is constant.
- If f is not constant, then the maximum of f is not an interior point.

2.2.13 Laurent expansion around an isolated singularity

Let $G \subset \mathbb{C}$ open, $z_0 \in G$, f analytic on $G \backslash \{z_0\}$. Then $\exists c_n \in \mathbb{C} (n \in \mathbb{Z})$ s.t. $\mathrm{B}(z_0, R) \subset G \Rightarrow f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \ \forall\ 0 < |z - z_0| < R$.

Useful Expansions:

- Maclaurin series for e^z is
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$
- Maclaurin series for $\sin(z)$ is
$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$
- Maclaurin series for $\cos(z)$ is
$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...$$

3 Know the PROOFS

- A continuous function on a compact set is uniformly continuous.
- Cauchy's Thm for a triangle (only the case $p \notin \Delta(a, b, c)$)
- Cauchy's Thm for starlike sets
- Cauchy's Integral Formula for star-like sets
- Liouville's Thm
- Morera's Thm