5 The equation with lower order terms – variable coefficients

We study the corner singularities in the solution of a differential equation with lower order terms. For this, we consider the Dirichlet problem in a sector S:

$$Lu := -\Delta u + p_1(x)u_{x_1} + p_2(x)u_{x_2} + q(x)u = f, \text{ in } S, \tag{1a}$$

$$u(x_1, 0) = g_0(x_1), \; x_1 > 0, \tag{1b}$$

$$u(r\cos\theta, r\sin\theta) = g_1(r), \; r > 0. \tag{1c}$$

We suppose that the coefficients p_1, p_2, and q are smooth. We also suppose that the problem (1a,b,c) is well-posed; that is, for each $f \in L_2(S)$ and $\{g_0, g_1\} \in H^{1/2}(\Gamma)$ there is a unique $u \in H^1(S)$ satisfying (1a,b,c) in the weak sense, and $\|u\|_{1,S} \leq C(\|f\|_{0,S} + \|\{g_0, g_1\}\|_{1/2,\Gamma}$. We are concerned with solutions $u \in H^1(S)$ of (1) which also satisfy

$$u(x) \equiv 0 \text{ for } r = \sqrt{x_1^2 + x_2^2} > 1. \tag{2}$$

The essential difference between (1a) and (2.2;1a) is that the lower order terms in (1a) contain singularities, and this introduces new linear functionals and new singular functions into the singular expansion of the solution.

In §2.2, there is derived a collection of linear functionals Λ_j on $\mathcal{Y}_D^s$ with the property that if $\{f, g_0, g_1\} \in \mathcal{Y}_D^s$ and if $\Lambda_j\{f, g_0, g_1\} = 0$ for $j = 1, \cdots, J(s)$, then the solution u of (2.2;1) belongs to $H^s(S)$. We seek to define a set of linear functionals $\Lambda_{L,j}$ associated with the problem (1) with similar properties. We shall use the data space $\mathcal{Y}_D^s$ defined in §2.2. As in §.2, we set $J(s) = \lfloor (s-1)/\alpha \rfloor$, and we define numbers $s_j = j\alpha + 1$. If $(s-1)/\alpha \neq$ integer, then $s \in (s_{J(s)}, s_{J(s)+1})$.

The definition of linear functionals $\Lambda_{L,j}$ associated with the operator L is a little tricky. Let $\{f, g_0, g_1\} \in \mathcal{Y}_D^s$ be given, and let u be a weak solution of (1). We would like to define a collection of linear functionals $\Lambda_{L,j}\{f, g_0, g_1\}$ by the formula

$$\Lambda_{L,j}\{f, g_0, g_1\} = \Lambda_j\{h, g_0, g_1\}, \; h = f - p_1 u_x - p_2 u_y - qu. \tag{3}$$

The right side of this equation might not be defined, since it might happen that $u \notin H^s(S)$, so $\{h, g_0, g_1\} \notin \mathcal{Y}_D^s$. The definition must be made in a recursive fashion. In this connection, we will say that $\Lambda_{L,j}\{f, g_0, g_1\}$ is *well-defined* if $\{h, g_0, g_1\} \in \mathcal{Y}_D^t$ for some $t > s_j$.

Lemma 1. *Let $s \geq 2$ with $(s-1)/\alpha \neq$ integer. Let $\{f, g_0, g_1\} \in \mathcal{Y}_D^s$, let $u \in H^1(S)$ be a solution of (1), and suppose u satisfies (2). (i) If $J(s) = 0$, then $u \in H^s(S)$. (ii) If $J(s) > 0$, then $u \in H^t(S)$ for any $t < s_1$. (iii) If $J(s) > 0$ and $u \in H^t(S)$ for any $t < J(s)$, then $\Lambda_{L,J(s)}\{f, g_0, g_1\}$ is well-defined. (iv) $u \in H^s(S)$ if and only if the functionals $\Lambda_{L,k}\{f, g_0, g_1\}$, $k = 1, 2, \cdots, J(s)$ are well-defined and vanish.*

Proof. (i) Suppose first that $J(s) = 0$, that is, $s < s_1$. We have $h = f - p_1 u_x - p_2 u_y - qu \in H^0(S)$, so $\{h, g_0, g_1\} \in \mathcal{Y}_D^2$. Since $J(2) \leq J(s) = 0$, we may apply Theorem 2.2.1;2 to conclude that $u \in H^2(S)$. Hence $h \in H^1(S)$, so $\{h, g_0, g_1\} \in \mathcal{Y}_D^{s^*}$ with $s^* = \min\{s, 3\}$. Since $J(s^*) \leq J(s) = 0$, we may apply Theorem 2.2.1;2 to conclude that $u \in H^{s^*}(S)$. Continuing in this manner, we obtain $u \in H^s(S)$.

(ii) There are two cases. Suppose $s_1 \leq 2$, and let $t < s_1$. Since $h \in H^0(S)$, $\{h, g_0, g_1\} \in \mathcal{Y}_D^t$, so from Theorem 2.2.1;2, $u \in H^t(S)$. Next, suppose $s_1 > 2$. Since $h \in H^0(S)$, $\{h, g_0, g_1\} \in \mathcal{Y}_D^2$, so $u \in H^2(S)$. Hence $h \in H^1(S)$, so $\{h, g_0, g_1\} \in \mathcal{Y}_D^{s^*}$ with $s^* = \min\{s, 3\}$. If $s^* = 3 < s_1$, then $\{h, g_0, g_1\} \in \mathcal{Y}_D^3$, so $u \in H^3(S)$, so $\{h, g_0, g_1\} \in \mathcal{Y}_D^{s^*}$ with $s^* = \min\{s, 4\}$. Continuing in this way, we get $u \in H^t(S)$ for any $t < s_1$.

(iii) Suppose $u \in H^t(S)$ for any $t < J(s)$. Then $h \in H^t(S)$ for any $t < J(s) - 1$. Since $\{g_1, g_2\} \in \mathcal{T}^s$ and $J(s) < s$, $\{h, g_0, g_1\} \in \mathcal{Y}^t$ for any $t < \min\{J(s) + 1, s\}$. Hence $\Lambda_{L,J(s)}\{h, g_0, g_1\}$ is well-defined.

(iv) If $u \in H^s(S)$, it is immediate that $\Lambda_{L,k}\{f, g_0, g_1\}$ is well-defined for $k = 1, \cdots, J(s)$, and from Theorem 2.2.1;2, $\Lambda_{L,k}\{f, g_0, g_1\} = 0$ for $k = 1, \cdots, J(s)$. Suppose $\Lambda_{L,k}\{f, g_0, g_1\}$ are well-defined and vanish for $k = 1, \cdots, J(s)$. From (ii), $u \in H^t(S)$ for any $t < s_1$. We show that, for each $k \leq J(s)$, $u \in H^t(S)$ for any $t < s_k$. The assertion is true for $k = 1$; suppose it is true for $k-1$. Thus, $u \in H^t(S)$ for any $t < s_{k-1}$. Hence $h \in H^t(S)$ for any $t < s_{k-1} - 1$, so $\{h, g_0, g_1\} \in \mathcal{Y}^t_D$ for any $t < s^* = \min\{s_{k-1} + 1, s_k\} \leq \min\{s_{k-1} + 1, s\}$. Since $\Lambda_{L,k-1}\{f, g_0, g_1\} = 0$, Theorem 2.2.1;2 implies that $u \in H^t(S)$ for any $t < s^*$. If $s^* = s_k < s_{k-1}$, we are finished with the inductive step. Otherwise, we know that $u \in H^t(S)$ for any $t < s_{k-1} + 1$, so $h \in H^t(S)$ for any $t < s_k$, so $\{h, g_0, g_1\} \in \mathcal{Y}^t_D$ for any $t < s^* = \min\{s_{k-1} + 2, s_k\}$. Continuing in this way, we find that $u \in H^t(S)$ for any $t < s_k$, which finishes the induction. Hence $u \in H^t(S)$ for any $t < s_{J(s)}$. The above argument now shows that $u \in H^t(S)$ for any $t < s^* = \min\{s_{J(s)} + 1, s\}$, with equality allowed in $s^* = s$. Continuing in this way, we eventually obtain $u \in H^s(S)$. ■

As an example of this theorem, consider the singular function v_j defined in (2.2.1;16). Letting S_1 denote the truncated sector of radius 1, the function $v_j \in H^s(S_1)$ for any $s < s_j$, and $v_j \notin H^{s_j}(S_1)$. Furthermore, $\Delta v_j = 0$ in S and $v_j = 0$ on ∂S. Let $\chi(r)$ be a smooth function which $\equiv 1$ in $(0, \frac{1}{2})$ and which $\equiv 0$ in $(1, \infty)$. Let $\bar{v}_j = \chi v_j$. Then

$$\bar{v}_j \in H^s(S) \text{ for } s < s_j,\ \bar{v}_j \notin H^{s_j}(S),\ L\bar{v}_j \in H^s(S) \text{ for } s < s_j - 1. \tag{4}$$

Hence $\{L\bar{v}_j, 0, 0\} \in \mathcal{Y}^s_D$ for $s < s_j + 1$. Applying Lemma 1, we find that $\Lambda_{L,k}\{L\bar{v}_j, 0, 0\}$ is well defined for $k = 1, \cdots, j$, $\Lambda_{L,k}\{L\bar{v}_j, 0, 0\} = 0$ for $j = 1, \cdots, j-1$, and $a_j = \Lambda_{L,j}\{L\bar{v}_j, 0, 0\} \neq 0$. If $\alpha < 1$, then $s_{j+1} < s_j + 1$, so $\Lambda_{L,j+1}\{L\bar{v}_j, 0, 0\}$ is well-defined. One could construct coefficients p_1, p_2, and q, so that $\Lambda_{L,j+1}\{L\bar{v}_j, 0, 0\} = 0$. In such a case, the vanishing of the single linear functional $\Lambda_{L,j+1}\{L\bar{v}_j, 0, 0\}$ does *not* imply that $\bar{v}_j$ belongs to $H^s(S)$ for $s_{j+1} < s < \min\{s_{j+2}, s_j + 1\}$ since, as we have seen in (4), $\bar{v}_j \notin H^{s_j}(S)$. The regularity result in Lemma 1 requires that *all* the functionals $\Lambda_{L,k}\{f, g_0, g_1\}$, $k = 1, \cdots, J(s)$, must vanish in order that $u \in H^s(S)$.

We now derive an expansion of a solution u of (1), (2) into a sum of singular functions plus a smooth remainder. To prepare for this expansion, we establish the following result.

Lemma 2. *Let β be real, let $l \geq 0$ be an integer, and let $\Theta(\theta)$ be a given smooth function. The problem*

$$\Delta u = r^\beta (\ln r)^l \Theta(\theta) \text{ in } S, \tag{5a}$$

$$u = 0 \text{ on } \Gamma, \tag{5b}$$

has a solution u of the form

$$u = \sum_{j=0}^{l+1} r^{\beta+2} (\ln r)^j \Psi_j(\theta) \tag{6}$$

where the Ψ_j are smooth functions satisfying $\Psi_j(0) = \Psi_j(\omega) = 0$, $j = 0, \cdots, l+1$. One has $\Psi_{l+1} = 0$ except in the case $(\beta + 2)/\alpha =$ integer.

Proof. Let $v = r^{\beta+2}(\ln r)^j \Psi(\theta)$, and note the formula

$$\begin{aligned} \Delta v =& \big[\Psi'' + (\beta+2)^2 \Psi\big] r^\beta (\ln r)^j \\ &+ \big[2j(\beta+2)(\ln r)^{j-1} + j(j-1)(\ln r)^{j-2}\big] r^\beta \Psi. \end{aligned} \tag{7}$$

There are two cases.

(i) Suppose $(\beta+2)/\alpha \neq$ integer. Define $u_l = r^{\beta+2}(\ln r)^l \Psi_l(\theta)$, where Ψ_l solves the problem

$$\Psi_l'' + (\beta+2)^2\Psi_l = \Theta,\ \Psi_l(0) = \Psi_l(\omega) = 0. \tag{8}$$

Because of our assumption, $(\beta+2)^2$ is not an eigenvalue, so the problem (8) has a unique solution. Since $\Delta(u-u_l) = r^\beta(\ln)^l\Theta +$ a linear combination of functions with lower powers on $\ln r$, the process may be repeated and eventually leads to (6). Note that in this case, l is the highest power of $\ln r$ that appears in the sum.

(ii) Suppose $\beta+2 = m\alpha$ with $m =$ integer. Thus, $\sin m\alpha\theta$ vanishes at $\theta = 0$ and $\theta = \omega$. Let

$$\begin{aligned} u_{l+1} &= r^{\beta+2}(\ln r)^{l+1}\sin m\alpha\theta, \\ u_l &= r^{\beta+2}(\ln r)^l \Psi_l(\theta), \end{aligned}$$

where the function Ψ_l is to be determined. From (7),

$$\Delta u_{l+1} = \big[2m\alpha(l+1)(\ln r)^l + l(l+1)(\ln r)^{l-1}\big] r^\beta \sin m\alpha\theta,$$

so

$$\begin{aligned} \Delta(u_l + cu_{l+1}) =& \big[\Psi_l'' + (m\alpha)^2\Psi_l + 2m\alpha(l+1)c\sin m\alpha\theta\big](\ln r)^l r^\beta \\ &+ \big[2m\alpha l\Psi_l + cl(l+1)\sin m\alpha\theta\big] r^\beta(\ln r)^{l-1} + l(l-1)\Psi_l(\ln r)^{l-2}r^\beta. \end{aligned}$$

We wish to choose the function $\Psi_l(\theta)$ and the constant c so that

$$\Psi_l'' + (m\alpha)^2\Psi_l = \Theta - 2m\alpha(l+1)c\sin m\alpha\theta, \tag{9a}$$

$$\Psi_l(0) = \Psi_l(\omega) = 0. \tag{9b}$$

To find Ψ_l and c, let $\Psi_{l,p}$ be a particular solution of (9a), and note the formula

$$\left\{\frac{d}{d\theta^2} + (m\alpha)^2\right\}\theta\cos m\alpha\theta = -2m\alpha\sin m\alpha\theta.$$

Let

$$\Psi_l(\theta) = \Psi_{l,p}(\theta) + c(l+1)\theta\cos m\alpha\theta + d\cos m\alpha\theta.$$

Then Ψ_l satisfies (9a), and with a proper choice of the coefficients c and d, Ψ_l also satisfies (9b). Since $\Delta(u - u_l - cu_{l+1}) = r^\beta(\ln)^l\Theta +$ a linear combination of functions with lower powers of $\ln r$, the process may be repeated and eventually leads to (6). ■

We now consider the expansion of a solution into singular functions. Let $\{f, g_0, g_1\} \in \mathcal{Y}_D^s$, let $u \in H^1(S)$ be a solution of (1), and suppose u satisfies (2). First, suppose $J(s) = 0$. Lemma 1 implies that $u \in H^s(S)$. In this case, u has the regularity dictated by the data $\{f, g_0, g_1\}$, and there is no expansion of u into singular functions plus a smoother remainder.

Now suppose $J(s) > 0$. From Lemma 1, $u \in H^t(S)$ for each $t < s_1$. Then $h \in H^t(S)$ for each $t < s_1 - 1$, so $\{h, g_0, g_1\} \in \mathcal{Y}_D^t$ for each $t < s^* = \min\{s, s_1+1\}$, and with equality if $(s_1+1)/\alpha \neq$ integer or if $s < s_1+1$. Let j^* be the largest integer such that $s_{j^*} < s^*$. Thus, if $(s_1+1)/\alpha \neq$ integer, $j^* = J(s^*)$, while if $s_1+1 = m\alpha$

for some integer m, $j^* = m-1$. By Lemma 1, (iii), $\Lambda_{L,j}\{f, g_0, g_1\}$ is well-defined for $j = 1, \cdots, j^*$. Using Theorem 2.2.1;2, we write

$$u = \sum_{j=1}^{j^*} \Lambda_{L,j}\{f, g_0, g_1\} v_j + w, \tag{10}$$

where $w \in H^t(S_a)$ for each truncated sector S_a and for each $t < s^*$. If $(s_1+1)/\pi \neq$ integer, $w \in H^{s^*}(S_a)$. The function v_j is given by (2.2.1;16); v_j is harmonic, vanishes on ∂S, and, for each truncated sector S_a, $v_j \notin H^{s_j}(S_a)$ and $v_j \in H^t(S_a)$ for $t < s_j$.

If $s = s^*$, and $(s_1+1)/\alpha \neq$ integer, then $w \in H^s(S_a)$ and (10) is the desired expansion of the solution into singular functions. In this case, the lower order terms in the operator L have introduced no new ingredients into the expansion.

Now suppose $s^* < s$. If $\Lambda_{L,1}\{f, g_0, g_1\} \neq 0$, then $u \notin H^{s_1}(S_a)$ and $h \notin H^{s_1-1}(S_a)$. This causes a difficulty in extending the expansion (10) to higher values of j. Lemma 2 is used to overcome this difficulty. For this, we rewrite (10) in a fashion that prepares for a recursion. Let $\mathcal{J}(s)$ be the set of pairs (j,i) of integers with $j \geq 1$, $i \geq 0$, and $j\alpha + i + 1 = s_j + i < s$. Let $v_{j,i,\mu}(x)$ be a function of the form

$$v_{j,i,\mu}(x) = r^{j\alpha+i}(\ln r)^\mu \Psi_{j,i,\mu}(\theta), \tag{11}$$

where $\Psi_{j,i,\mu}(\theta)$ is a smooth function of θ. We write the expansion (10) as

$$u = \sum_{(j,i)\in\mathcal{J}(s^*)} \sum_{\mu=0}^{1} \Lambda_{L,j,i,\mu}\{f, g_0, g_1\} v_{j,i,\mu} + w. \tag{10$'$}$$

From the definition of $\mathcal{J}$, $(j,i) \in \mathcal{J}(s^*)$ implies $i = 0$, so (10′) is the same as (10) with a proper definition of the linear functionals $\Lambda_{L,j,i,\mu}$ and functions $v_{j,i,\mu}$. Comparing with (2.2;16), we see that the factor $\ln r$ appears only if $j\alpha$ is an integer.

To prepare for the recursive definition of our expansion into singular functions, we establish

Lemma 3. *Let $s \geq 2$ with $(s-1)/\alpha \neq$ integer. Let $\{f, g_0, g_1\} \in \mathcal{Y}_D^s$, let $u \in H^1(S)$ be a solution of (1), and suppose u satisfies (2). Suppose $s^* < s$, and suppose $u = V + w$ with $w \in H^t(S_a)$ for each $a > 0$ and each $t < s^*$, and with*

$$V = \sum_{(j,i)\in\mathcal{J}(s^*)} \sum_{\mu=0}^{m} \Lambda_{L,j,i,\mu}\{f, g_0, g_1\} v_{j,i,\mu}, \tag{12}$$

where $\Lambda_{L,j,i,\mu}$ are bounded linear functionals of $\mathcal{Y}_D^t$ for each $t < s^*$. Let $s_1^* = \min\{s, s^*+1\}$. Then $u = V_1 + w_1$ where $w_1 \in H^t(S_a)$ for each $a > 0$ and each $t < s_1^*$, and where

$$V_1 = \sum_{(j,i)\in\mathcal{J}(s_1^*)} \sum_{\mu=0}^{m+1} \Lambda^{(1)}_{L,j,i,\mu}\{f, g_0, g_1\} v^{(1)}_{j,i,\mu}, \tag{13}$$

where $v^{(1)}_{j,i,\mu}$ are functions of the form (11) with $\Psi_{j,i,\mu}$ replaced by $\Psi'_{j,i,\mu}$, and the $\Lambda^{(1)}_{L,j,i,\mu}$ are bounded linear functionals on $\mathcal{Y}_D^t$ for each $t < s_1^*$. If the linear functionals $\Lambda_{L,j,i,\mu}$ depend continuously on p_1, p_2 and q, regarded as elements of the Banach space $C^{\lceil s\rceil-2}(S)$, then the linear functionals $\Lambda^{(1)}_{L,j,i,\mu}$ have the same property. If α is irrational, then in the sum over μ in (13), the upper limit $m+1$ may be replaced by m.

Proof. Define an operator T by $Tw = p_1 w_{x_1} + p_2 w_{x_2} + qw$, so $Lw = -\Delta w + Tw$. For any $\delta > 0$,

$$\begin{aligned} &v_{j,i,\mu} \in H^{s_j+i-\delta}(S_a),\ x_1^k x_2^{m-k} v_{j,i,\mu} \in H^{s_j+i+m-\delta}(S_a), \\ &D_x v_{j,i,\mu} \in H^{s_j+i-1-\delta}(S_a),\ x_1^k x_2^{m-k} D_x v_{j,i,\mu} \in H^{s_j+i+m-1-\delta}(S_a); \end{aligned} \tag{14}$$

furthermore, these inclusions do not hold for $\delta = 0$. In (14), D_x denotes $\partial/\partial x_1$ or $\partial/\partial x_2$. Choose $m = m_{j,i}$ so that $s_j + i + m_{j,i} > s - 2$, and expand each of the functions p_1 and p_2 in a Taylor series T of degree $m_{j,i}$ with remainder R. Similarly, expand q in a Taylor series T of degree $m_{j,i} - 1$ with remainder R. We write these expansions as

$$\begin{aligned} p_1 &= T(p_1; x) + R(p_1; x), \\ p_2 &= T(p_2; x) + R(p_2; x), \\ q &= T(q; x) + R(q; x). \end{aligned}$$

Since $|R(p_1; x)| \le Cr^{m_{j,i}+1}$, (14) implies that $R(p_1; x) D_{x_1} v_{j,i,\mu} \in H^{s-2}(S_a)$, and similarly for the other coefficients. Writing the Taylor polynomials as a sum of homogeneous polynomials of degree $\le m_{j,i}$, expressing the result in polar coordinates, and inserting the expansions into Tu, we obtain

$$Tv_{j,i,\mu} = \sum_{k=0}^{m_{j,i}} \sum_{\nu=\mu-1}^{\mu} r^{j\alpha+i-1+k} (\ln r)^{\nu} \Theta_{j,i,k,\nu}(\theta) + R,$$

where $R \in H^s(S_a)$ and where the $\Theta_{j,i,k,\nu}$ are smooth functions of θ. Using Lemma 2, we obtain functions $v_{j,k,\nu}^{(j,i,\mu)}$ of the form (11) such that

$$\Delta \sum_{k=1}^{m_{j,i}+1} \sum_{\nu=0}^{\mu+1} v_{j,k,\nu}^{(j,i,\mu)} = Tv_{j,i,\mu} - R. \tag{15}$$

Set

$$\begin{aligned} Z_1 &= \sum_{(j,i)\in\mathcal{J}(s^*)} \sum_{\mu=0}^{m} \Lambda_{L,j,i,\mu}\{f, g_0, g_1\} \sum_{k=1}^{m_{j,i}+1} \sum_{\nu=0}^{\mu+1} v_{j,k,\nu}^{(j,i,\mu)} \\ &= \sum_{(j,i)\in\mathcal{J}(s_1^*)} \sum_{\mu=0}^{m+1} \Lambda_{L,j,i,\mu}^{(1)}\{f, g_0, g_1\} v_{j,i,\mu}^{(1)} \end{aligned}$$

where the linear functionals $\Lambda_{L,j,i,\mu}^{(1)}$ are defined by combining all terms containing $r^{j\alpha+i}(\ln r)^{\mu}$. Then (15) yields $\Delta Z_1 = TV - R$. Let m^* denote the largest value of the $m_{j,i}$. This value comes with the smallest value of s_j and i. Thus, m^* must satisfy $s_1 + m^* > s$, or, $\alpha + m^* > s - 1$. If s is an integer, we may take $m^* = s - 1$. If $s \ne$ integer, we may take $m^* = \lceil s \rceil - 1$. In either case, we have $m^* = \lceil s \rceil - 1$. With this choice for m^*, the linear functionals $\Lambda_{L,j,i,\mu}^{(1)}$ appearing in Z_1 have the asserted dependence on p_1, p_2, and q.

Let $u_1 = u - Z_1$. Since Z_1 is already in the form of a sum of singular functions, to prove the result it suffices to make an expansion of u_1. We have

$$\begin{aligned} -\Delta u_1 &= -\Delta u + \Delta Z_1 \\ &= f - Tu + TV - R \\ &= f - TV - Tw + TV - R \\ &= f - Tw - R. \end{aligned}$$

Since $Z_1 = 0$ on $\partial\Omega$, $u_1 = g_l$ on Γ_l, $l = 0, 1$. Evidently $Tw \in H^t(S_a)$ for each $t < s^* - 1$, so $\{-\Delta u_1, g_0, g_1\} \in \mathcal{Y}_D^t$ for each $t < s_1^* = \min\{s, s^* + 1\}$. Since $s^* < s_1^*$, $\{-\Delta u_1, g_0, g_1\}$ is smoother than $\{f, g_0, g_1\}$, and we have the possibility of obtaining an expansion of u_1 with a smoother remainder than that occuring in the expansion of u. Let j_1^* be the largest integer such that $s_{j_1^*} < s^*$. We have from Theorem 2.2.1;2,

$$u_1 = \sum_{j=1}^{j_1^*} \Lambda_j\{-\Delta u_1, g_0, g_1\} v_j + w_1, \tag{16}$$

where $w_1 \in H^t(S_a)$ for each $t < s_1^*$. As in the rewriting of (10) in the form (10′), we write

$$Z_2 = \sum_{j=1}^{j_1^*} \Lambda_j\{-\Delta u_1, g_0, g_1\} v_j = \sum_{j=1}^{j_1^*} \sum_{\mu=0}^{1} \Lambda^{(2)}_{L,j,0,\mu}\{f, g_0, g_1\} v^{(2)}_{j,0,\mu}, \tag{16′}$$

where $v^{(2)}_{j,0,\mu}$ are functions of the form (11) and $\Lambda^{(2)}_{L,j,0,\mu}$ are bounded linear functionals on $\mathcal{Y}^s_D$. We then have

$$u = Z_1 + Z_2 + w_1,$$

and since $Z_1 + Z_2$ is of the form (13), we obtain the desired expansion $u = V_1 + w_1$. The linear functionals $\Lambda^{(2)}_{L,j,0,\mu}$ appearing in Z_2 depend continuously on p_1, p_2, and q, as elements of $C^0(S)$. If α is irrational, then $(j\alpha + i)/\alpha \neq$ integer, so in the application of Lemma 2, one is in the case $(\beta + 2)/\alpha \neq$ integer. Hence in (15), the upper limit $\mu + 1$ in the sum over ν can be replaced by μ. Also, in (16), the v_j do not involve $\ln r$, so in (16′) one has only the case $\mu = 0$. ■

Lemma 3 is used to obtain the following expansion theorem.

Theorem 1. *Let $s \geq 2$, and suppose $s \neq j\alpha + i$ for integers j and i. There are bounded linear functionals $\Lambda_{L,j,i,\mu}$ on $\mathcal{Y}^s_D$, $\{j, i\} \in \mathcal{J}(s)$, $\mu = 0, \cdots, \mu^*$, and smooth functions $\Psi_{j,i,\mu}(\theta)$, such that if $\{f, g_0, g_1\} \in \mathcal{Y}^s_D$ and if u is the corresponding solution of (1) which satisfies (2), then*

$$u = \sum_{\{j,i\} \in \mathcal{J}(s)} \sum_{\mu=0}^{\mu^*} \Lambda_{L,j,i,\mu}\{f, g_0, g_1\} r^{j\alpha+i} (\ln r)^\mu \Psi_{j,i,\mu}(\theta) + w,$$

with the remainder $w \in H^s(S_a)$. The integer μ^* satisfies $\alpha + \mu^* - 1 < s < \alpha + \mu^*$. The linear functionals $\Lambda_{L,j,i,\mu}$ and the remainder w depend continuously on p_1, p_2 and q in the Banach space $C^{\lceil s \rceil}(S)$, and they satisfy the inequalities

$$|\Lambda_{L,j,i,\mu}(f)| \leq C(K) ||f||_{s-2,S}, \tag{17}$$

$$||w||_{s,S_a} \leq C(a, K) ||f||_{s-2,S}, \tag{18}$$

where K denotes the suprema of p_1, p_2 and q and all their derivatives of order $\leq \lceil s \rceil$. If α is irrational, then we may take $\mu^* = 0$, and there are no terms involving $\ln r$ in the expansion. The solution $u \in H^s(S)$ if and only if $\Lambda_{L,j,i,\mu}\{f, g_0, g_1\} = 0$ for all $(j, i) \in \mathcal{J}(s)$ and $\mu = 0, \cdots, \mu^*$ such that $\Psi_{j,i,\mu}(\theta)$ is not identically 0.

Proof. Let s be given, and let i be the largest integer such that $s_1 + i < s$. The proof is by induction on i. If $i = 0$, then from (10) we have the desired representation of u, with $\mu^* = 1$. Suppose $i > 0$. Since $\{f, g_0, g_1\} \in \mathcal{Y}^s_D$, $\{f, g_0, g_1\} \in \mathcal{Y}^t_d$ for $t < s$. From the induction hypothesis, $u = V + w$ with $w \in H^{s^*}(S_a)$ with $s^* = s_1 + i$. Applying Lemma 3, we obtain $u = V_1 + w_1$ with $w_1 \in H^s(S_a)$. The formula for μ^* follows from the construction. In α is irrational, the formula (2.2;16) produces no factors of $\ln r$, and Lemma 2 introduces no factors of $\ln r$, so we may take $\mu^* = 0$. Since the functions $r^{j\alpha+i}(\ln r)^\mu$ are linearly independent for different values of j, i, and μ, we conclude that $u \in H^s(S_a)$ if and only if all the linear functionals vanish. The asserted dependence on the coefficients p_1, p_2, and q, and the inequalities (17) and (18), follow readily. ■