2.4.1 A singularly perturbed convection–diffusion problem

We now apply the results of Section 2.4 to a problem of particular interest: determining the singular functions and linear functionals associated with a singularly perturbed convection diffusion problem. Consider the problem

$$Lu := -\varepsilon\Delta u + p_1 u_{x_1} + p_2 u_{x_2} + qu = f, \text{ in } S, \tag{1a}$$

$$u = 0 \text{ on } \Gamma. \tag{1b}$$

Here, the coefficients of the operator L are assumed to be constant, the parameter $\varepsilon \in (0,1]$, and it is of particular interest if $\varepsilon << 1$. Depending on the direction of the vector $[p_1, p_2]^T$, the solution u can have interior or boundary layers.

Let $d = 1/(2\varepsilon)$ and set $u(x,y) = e^{d(p_1x_1+p_2x_2)}u_1(x,y)$. A computation shows that

$$Lu = \tfrac{1}{2}d[-4\varepsilon^2\Delta u_1 + (p_1^2 + p_2^2 + 4\varepsilon q)u_1]e^{d(p_1x_1+p_2x_2)}.$$

We will use "stretched" variables $\xi_1 = dx_1$, $\xi_2 = dx_2$, and we shall also write $\rho = (\xi_1^2 + \xi_2^2)^{1/2} = dr$. Setting $u_2(\xi_1, \xi_2) = u_1(x_1, x_2)$, we have

$$Lu = \tfrac{1}{2}d[-\Delta_{(\xi_1,\xi_2)}u_2 + (p_1^2 + p_2^2 + 4\varepsilon q)u_2]e^{p_1\xi_1+p_2\xi_2}.$$

Set $k^2 = (p_1^2 + p_2^2 + 4\varepsilon q)$. Motivated by Section 2.4 we seek a singular function for the problem (1a,b) in the form $\zeta_j(x,y) = e^{p_1\xi_1+p_2\xi_2}\zeta_{j,2}(\xi,\eta)$, where $\zeta_{j,2}(\xi,\eta) = R_j(\rho)\sin j\alpha\theta$. We have

$$-\Delta_{(\xi_1,\xi_2)}\zeta_{j,2} + k^2\zeta_{j,2} = (M_{j,\rho}R_j(\rho))\sin j\alpha\theta,$$

where $M_{j,\rho}$ denotes the differential operator of Section 4, but applied in the ρ variable. Let ψ_j be a smooth function such that $\psi_j = 0$ in $(0,1)$ and $\psi_j(r) = 0$ outside $(0,2)$. An additional condition on ψ_j is given below. Let $R_j(r)$ satisfy the two point boundary value problem

$$M_{j,\rho}R_{j,\rho} = \psi_j(r),\ R_j(0) = 0,\ R_j(r) \to 0 \text{ as } r \to \infty. \tag{2}$$

Having $R_j(\rho)$, and therefore $\zeta_{j,2}(\xi_1, \xi_2)$, we define

$$\zeta_j(x_1, x_2) = e^{p_1\xi_1+p_2\xi_2}\zeta_{j,2}(\xi,\eta).$$

We have

$$L\zeta_j = \tfrac{1}{2}de^{p_1\xi_1+p_2\xi_2}\psi_j(d\rho)\sin j\alpha\theta.$$

The equation $M_{j,\rho}R = 0$ is satisfied by the Bessel functions $I_{j\alpha}(k\rho)$ and $K_{j\alpha}(k\rho)$. Since $\psi_j(\rho) \equiv 0$ outside $1 < \rho < 2$, it follows that $R_j(\rho)$ is a linear combination of these two Bessel functions outside $(1,2)$. Because of the boundary conditions in (2) and the asymptotic properties (4.7), we have

$$\zeta_j(x_1, x_2) \sim r^{j\alpha} + O(r^{j\alpha+1}) \quad \text{near } r = 0, \tag{3a}$$

$$\zeta_j(x_1, x_2) \sim r^{-1}\exp\left\{-d\left([q + \tfrac{1}{4}(p_1^2 + p_2^2)r]^{1/2} - \tfrac{1}{2}(p_1x_1 + p_2x_2)\right)\right\} \quad \text{near } r = \infty. \tag{3b}$$

Note that the function $r^\beta \in H^s(S_1)$ if and only if $s < \beta + 1$. Hence, using (3a) and (3b), $\zeta_j \in H^s(S)$ if and only if $s < j\alpha + 1 = s_j$, which conforms with the assertion of Theorem 2.2.1;3 regarding v_j. Since

$-\Delta\zeta_j = L\zeta_j - p_1\zeta_{j,x_1} - p_2\zeta_{j,x_2} - q\zeta_j$ and $L\zeta_j$ is smooth, $-\Delta\zeta_j \in H^s(S)$ for $s < s_j - 1$. Hence $\Lambda_j''(-\Delta\zeta_j)$ is well-defined. To calculate $\Lambda_j''(-\Delta\zeta_j)$, we use the formula (2.2.1;17b). This formula requires the Taylor polynomial of degree $\lfloor j\alpha \rfloor - 2$ of $-\Delta\zeta_j$. Since $L\zeta_j \equiv 0$ near $r = 0$, its Taylor polynomial is zero. From (10a) it is seen that the Taylor polynomials of ζ_j, ζ_{j,x_1}, and ζ_{j,x_2} are zero. Hence the Taylor polynomial of $-\Delta\zeta_j$ is zero and we have

$$\Lambda_j''(-\Delta\zeta_j) = \frac{1}{j\pi}\int\int_S (-\Delta\zeta_j) r^{-j\alpha} \sin j\alpha\theta dx. \tag{4}$$

We now assert that

$$\Lambda_j''(-\Delta\zeta_j) = \frac{-C_j}{j\pi}\varepsilon^{-j\alpha-1}\int\int_S (L\zeta_j) e^{-(p_1\xi_1+p_2\xi_2)/2} K_{j\alpha}(k\rho) \sin j\alpha\theta dx, \tag{5}$$

where $C_j = k^{j\alpha}2^{-j\alpha}B_j^{-1}$ and B_j is the constant occuring in (4.7). In the proof of (5) we use the adjoint operator L^*, defined by

$$L^*u := -\varepsilon\Delta u - p_1 u_{x_1} - p_2 u_{x_2} + qu.$$

To show (5) write $\varepsilon\Lambda_j(-\Delta\zeta_j) = \Lambda_j(L\zeta_j) - \Lambda_j(p_1\zeta_{j,x_1} + p_2\zeta_{j,x_2} + q\zeta_j)$. Also, define

$$\tilde{K}(x,y) = e^{-(p_1\xi_1+p_2\xi_2)/2} K_{j\alpha}(k\rho).$$

We must show that

$$\int\int_S L\zeta_j\{r^{-j\alpha} - C_j\varepsilon^{-j\alpha}\tilde{K}\}\sin j\alpha\theta dx = \int\int_S r^{-j\alpha}\sin j\alpha\theta(p_1\zeta_{j,x_1} + p_2\zeta_{j,x_2} + q\zeta_j)dx. \tag{6}$$

For this we write

$$\begin{aligned}
&\int\int_S L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx \\
&\qquad = \lim_{\delta\to 0}\int\int_{S_\delta^c} L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx \\
&\qquad = \lim_{\delta\to 0}\int\int_{S_\delta^c} \zeta_j\{L^*(r^{-j\alpha}\sin j\alpha\theta) - C_j\varepsilon^{-j\alpha}L^*(\tilde{K}\sin j\alpha\theta)\})dx \\
&\qquad\quad - \varepsilon\lim_{\delta\to 0}\int_{r=\delta}\zeta_{j,r}\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}\delta d\theta \\
&\qquad\quad + \varepsilon\lim_{\delta\to 0}\int_{r=\delta}\zeta_j\{-j\alpha r^{-j\alpha-1}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}_r\sin j\alpha\theta\}\delta d\theta \\
&\qquad\quad - \lim_{\delta\to 0}\int_{r=\delta}(p_1\cos\theta + p_2\sin\theta)\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}(x,y)\sin j\alpha\theta\}\delta d\theta.
\end{aligned}$$

As in the calculation that leads to (4.6a,b), one sees that $L^*(\tilde{K}\sin j\alpha\theta\}) = 0$. Also, using (4.7) and (3a) we see that the integrals over $r = \delta$ tend to zero as $\delta \to 0$. We therefore get

$$\int\int_S L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx = \lim_{\delta\to 0}\int\int_{S_\delta^c}\zeta_j\{L^*(r^{-j\alpha}\sin j\alpha\theta)dx.$$

Since $r^{-j\alpha}\sin j\alpha\theta$ is a harmonic function, $L^*(r^{-j\alpha}\sin j\alpha\theta) = -p_1(r^{-j\alpha}\sin j\alpha\theta)_{x_1} - p_2(r^{-j\alpha}\sin j\alpha\theta)_{x_2} + qr^{-j\alpha}\sin j\alpha\theta$. Hence

$$\begin{aligned}
\int\!\!\int_S & L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx \\
&= \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c}\zeta_j\{-p_1(r^{-j\alpha}\sin j\alpha\theta)_{x_1} - p_2(r^{-j\alpha}\sin j\alpha\theta)_{x_2} + qr^{-j\alpha}\sin j\alpha\theta)\}dx \\
&= \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c}(p_1\zeta_{j,x_1} + p_2\zeta_{j,x_2} + q\zeta_j)(r^{-j\alpha}\sin j\alpha\theta)dx \\
&\quad + \lim_{\delta\to 0}\int_{r=\delta}(p_1n_1 + p_2n_2)\zeta_j r^{-j\alpha}\sin j\alpha\theta)\delta d\theta.
\end{aligned}$$

Again it is seen that the integral over $r = \delta$ tends to zero as $\delta \to 0$. This completes the proof of (6), and hence of (5).

Using (5) we have

$$\Lambda_j''(-\Delta\zeta_j) = -\frac{C_j}{4j\pi}\varepsilon^{-j\alpha-2}\int\!\!\int_S \psi_j(\rho)K_{j\alpha}(k\rho)\sin^2 j\alpha\theta dx. \tag{7}$$

The function ψ_j is chosen so that the integral in (7) is non-zero, and is normalized so that

$$\int\!\!\int_S \psi_j(\rho)K_{j\alpha}(k\rho)\sin^2 j\alpha\theta dx = -4j\pi C_j^{-1}\varepsilon^{j\alpha+2}. \tag{8}$$

Then $\Lambda_j(-\Delta\zeta_j) = 1$.

For the following theorem we let $A : f \mapsto u$ denote the solution operator to (1a,b), and we write $Nf = \varepsilon^{-1}[f - p_1(Af)_{x_1} - p_2(Af)_{x_2} - qAf$. Thus, $Lu = f$ implies $-\Delta u = Nf$. We shall also use the fact that Lemma 4.1 applies to the problem (1a,b).

Theorem 1. *There are linear functionals $\Lambda_{L,j}$ and functions ζ_j, $j = 1, 2, \cdots$, with the following properties. (i) $\Lambda_{L,j}$ is a bounded linear functional on $H^{s-2}(S)$ for $s > s_j$, but not for $s \le s_j$. (ii) $\zeta_j \in H^s(S) \cap H_0^1(S)$ for $s < s_j$ but not for $s \ge s_j$. Also ζ_j is smooth everywhere except at the origin, $L\zeta_j = 0$ in a neighborhood of the origin, and $L\zeta_j = 0$ for large r. (iii) For some $j \ge 0$, let $s_j < s < s_{j+1}$, $f \in H^{s-2}(S)$ and $u = Af$. Then*

$$u_j := u - \sum_{l=1}^{j}\Lambda_{L,l}(f)\zeta_l \in H^s(S), \quad \textit{with } \|u_j\|_s \le C\|f\|_{s-2}. \tag{9}$$

If $j = 0$, the sum is replaced by 0, $u \in H^s(S)$, and $\|u\|_s \le C\|f\|_{s-2}$.

Proof. The proof is given in a series of steps.

(a). Let χ be a smooth function which is $\equiv 1$ in a neighborhood of the origin and $\equiv 0$ for $r \ge 1$. Since $u \in H^1(S)$, $\chi u \in H^1(S)$ and $L(\chi u) \in H^{\min(s-2,0)}(S)$. Hence $NL(\chi u) \in H^{\min(s-2,0)}(S)$. Lemma 4.1 implies that $\chi u \in H^s(S)$. Since $(1-\chi)u$ solves an elliptic problem on a smooth domain, it can be shown that $(1-\chi)u \in H^s(S)$. Hence $u \in H^s(S)$. This gives the decomposition (9) in the case $j = 0$.

(b). Suppose that $s_1 < s < s_2$. From (a), $u \in H^{s'}(S)$ for any $s' < s_1$. Hence $\chi u \in H^{s'}(S)$. Hence $F = NL(\chi u) \in H^{s'-2}(S)$ for any $s' < s_1 + 1$. Pick a number $s' > s_1$ with $s' < \min(s, s_1 + 1, s_2)$. Then $\Lambda_1(F)$ is well-defined. Define $\Lambda_{L,1}(f) = \Lambda_1''(F)$. Let $u_1^* = \chi u - \Lambda_{L,1}(f)\zeta_1$. Then $\Lambda_{L,1}(Lu_1^*) = \Lambda_1''(-\Delta u_1^*) = 0$ so from Theorem 2.1.1;3, $u_1^* \in H^{s'}(S)$. Also $Lu_1^* \in H^{s-2}(S)$. Applying Lemma 4.1, $u_1^* \in H^s(S)$. Since

$(1-\chi)u$ solves an elliptic problem on a smooth domain, it can be shown that $(1-\chi)u \in H^s(S)$. Setting $u_1 = u_1^* + (1-\chi)u$, we have $u = \Lambda_{L,1}(f)\zeta_1 + u_1$ with $u_1 \in H^s(S)$. This gives the decomposition (9) in the case $j = 1$.

(c) We use an argument by induction and a recursive definition of the linear functionals. Suppose the linear functionals $\Lambda_{L,i}$ have been defined for $i < j$. Let $s_j < s < s_{j+1}$, let $f \in H^{s-2}(S)$, and let u be the solution of (1a,b). By induction, we may write $u = \sum_{l=1}^{j-1} \Lambda_{L,l}(f)\zeta_l + u_{j-1}$ with $u_{j-1} \in H^{s'}(S)$ for any $s' < s_{j-1}$. Hence $\chi u_{j-1} \in H^{s'}(S)$. Hence $F = NL(\chi u_{j-1}) \in H^{s'-2}(S)$ for any $s' < s_{j-1}+1$. Pick a number $s' > s_{j-1}$ with $s' < \min(s, s_{j-1}+1, s_2)$. Then $\Lambda_j''(F)$ is well-defined. We define $\Lambda_{L,j}(f) = \Lambda_j''(F)$. Let $u_j^* = \chi u_{j-1} - \Lambda_{L,j}(f)\zeta_j$. Then $\Lambda_{L,j}(Lu_j^*) = \Lambda_j''(-\Delta u_j^*) = 0$ so from Theorem 2.1.1;3, $u_j^* \in H^{s'}(S)$. Also $Lu_j^* \in H^{s-2}(S)$. Applying Lemma 4.1, $u_j^* \in H^s(S)$. Since $(1-\chi)u_{j-1}$ solves an elliptic problem on a smooth domain, it can be shown that $(1-\chi)u_{j-1} \in H^s(S)$. Setting $u_j = u_j^* + (1-\chi)u_{j-1}$, we have $u = \Lambda_{L,j}(f)\zeta_j + u_j$ with $u_j \in H^s(S)$. This gives the decomposition (9) in the general case. ■

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We are now concerned with calculating the linear functional $\Lambda_{L,j}$. The following lemmas gives some information on this.

Lemma 1. *Let* $f \in H^{s-2}(S)$ *for* $s > s_j - 2 = j\alpha - 1$, *so* $f \in C^{\lfloor j\alpha - 2\rfloor}(S)$. *Suppose* $\Lambda_{L,l}(f) = 0$ *for* $l = 1, \cdots, j-1$, *so* $u = Af \in H^s(S)$ *for* $s < s_j$. *Suppose* u *and its derivatives of order* $\le \lfloor j\alpha \rfloor$ *vanish at the origin. Then*

$$\Lambda_{L,j}(f) = \frac{-C_j}{j\pi}\varepsilon^{-j\alpha-1}\int\int_S f(x)e^{-(p_1\xi_1+p_2\xi_2)/2}K_{j\alpha}(k\rho)\sin j\alpha\theta dx. \tag{10}$$

Proof. Since $u \in H^s(S)$ for $s < s_j = j\alpha + 1$, $u \in C^s(S)$ for $s < j\alpha$. Hence the vanishing condition in the statement of the lemma has meaning, and

$$|u(x)| + r|Du(x)| \le Cr^s \quad \text{for } s < j\alpha. \tag{11}$$

By definition,

$$\begin{aligned}
\Lambda_{L,j}(f) = \Lambda_j''(f) &= \frac{1}{j\pi}\int\int_S (-\Delta u) r^{-j\alpha}\sin j\alpha\theta dx \\
&= \frac{1}{j\pi\varepsilon}\int\int_S f(x) r^{-j\alpha}\sin j\alpha\theta dx \\
&\quad - \frac{1}{j\pi\varepsilon}\int\int_S [(p_1u_{x_1} + p_2u_{x_2} + qu)] r^{-j\alpha}\sin j\alpha\theta dx \\
&= -\frac{1}{j\pi\varepsilon}C_j\varepsilon^{-j\alpha}\int\int_S f(x)e^{-(p_1\xi_1+p_2\xi_2)/2}K_{j\alpha}(k\rho)\sin j\alpha\theta dx \\
&\quad - \frac{1}{j\pi\varepsilon}\int\int_S f(x)\{r^{-j\alpha} - C_j\varepsilon^{-j\alpha}\tilde{K}_{j\alpha}(k\rho)\}\sin j\alpha\theta dx \\
&\quad - \frac{1}{j\pi\varepsilon}\int\int_S [(p_1u_{x_1} + p_2u_{x_2} + qu)] r^{-j\alpha}\sin j\alpha\theta dx.
\end{aligned}$$

We must show that

$$\int\int_S Lu\{r^{-j\alpha} - C_j\varepsilon^{-j\alpha}\tilde{K}_{j\alpha}(k\rho)\}\sin j\alpha\theta dx = \int\int_S [(p_1u_{x_1} + p_2u_{x_2} + qu)] r^{-j\alpha}\sin j\alpha\theta dx.$$

For this we write

$$
\begin{aligned}
&\int\!\!\int_S Lu\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx \\
&\qquad = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} Lu\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx \\
&\qquad = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} u\{L^*(r^{-j\alpha}\sin j\alpha\theta) - C_j\varepsilon^{-j\alpha}L^*(\tilde{K}\sin j\alpha\theta)\})dx \\
&\qquad\quad - \varepsilon\lim_{\delta\to 0}\int_{r=\delta} u_r\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}\delta d\theta \\
&\qquad\quad + \varepsilon\lim_{\delta\to 0}\int_{r=\delta} u\{-j\alpha r^{-j\alpha-1}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}_r\sin j\alpha\theta\}\delta d\theta \\
&\qquad\quad - \lim_{\delta\to 0}\int_{r=\delta}(p_1\cos\theta + p_2\sin\theta)u\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}(x,y)\sin j\alpha\theta\}\delta d\theta.
\end{aligned}
$$

As in the calculation that leads to (4.6a,b), one sees that $L^*(\tilde{K}\sin j\alpha\theta\}) = 0$. Also, using (11) we see that the integrals over $r = \delta$ tend to zero as $\delta \to 0$. We therefore get

$$
\int\!\!\int_S Lu\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} u\{L^*(r^{-j\alpha}\sin j\alpha\theta)dx.
$$

Since $r^{-j\alpha}\sin j\alpha\theta$ is a harmonic function $L^*(r^{-j\alpha}\sin j\alpha\theta) = -p_1(r^{-j\alpha}\sin j\alpha\theta)_{x_1} - p_2(r^{-j\alpha}\sin j\alpha\theta)_{x_2} + qr^{-j\alpha}\sin j\alpha\theta$. Hence

$$
\begin{aligned}
&\int\!\!\int_S Lu\{r^{-j\alpha}\sin j\alpha\theta - C_j\varepsilon^{-j\alpha}\tilde{K}\sin j\alpha\theta\}dx \\
&\qquad = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} u\{-p_1(r^{-j\alpha}\sin j\alpha\theta)_{x_1} - p_2(r^{-j\alpha}\sin j\alpha\theta)_{x_2} + qr^{-j\alpha}\sin j\alpha\theta)\}dx \\
&\qquad = \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c}(p_1u_{x_1} + p_2u_{x_2} + qu)(r^{-j\alpha}\sin j\alpha\theta)dx \\
&\qquad\quad + \lim_{\delta\to 0}\int_{r=\delta}(p_1n_1 + p_2n_2)ur^{-j\alpha}\sin j\alpha\theta)\delta d\theta.
\end{aligned}
$$

Again it is seen that the integral over $r = \delta$ tends to zero as $\delta \to 0$. This completes the proof of (10). ■

Lemma 2. *Let $u = Af$. Suppose that $u \in C^m(S_1)$, $f \in C^n(S_1)$, for $n > ***m$. Then $|D^ju(x)| \le C$, $x \in S_1$, with C independent of ε.*