4 The equation with lower order terms – constant coefficients

In earlier sections we have derived a singularity expansion for the operator $-\Delta$. We now consider a problem with lower order terms. Thus, we shall consider in a sector S the problem

$$Lu := -\Delta u + p_1 u_{x_1} + p_2 u_{x_2} + qu = f, \text{ in } S, \tag{1a}$$

$$u = 0 \text{ on } \Gamma. \tag{1b}$$

Our procedure will be to put the lower order terms on the right side of the equation and then apply the results of the earlier analysis. However the new right hand side now has some singularities, because it depends on the solution which has singularities. This introduces extra singular functions into the singularity expansion of the solution of (1a,b). In the case of constant coefficients, which is the case considered in this section, an alternate procedure can be used which eliminates the need for these extra functions. The present section describes this, while the following section considers the case of lower order terms with variable coefficients.

In this section, we consider the problem (1a,b) with the assumptions that p_1, p_2, and q are constants with $q > 0$. Using the fact that $q > 0$, an energy argument shows that if $f \in L_2(S)$ or $H^{-1}(S)$, then (1a,b) has a weak solution $u \in H_0^1(S)$. We denote by $A : f \mapsto u$ the solution operator to the problem (1a,b). A is a bounded map from $H^{-1}(S)$ to $H_0^1(S)$. Our approach is to write the problem in the form

$$-\Delta u = F := f - p_1 u_{x_1} - p_2 u_{x_2} - qu, \text{ in } S, \tag{2a}$$

$$u = 0 \text{ on } \Gamma \tag{2b}$$

and then apply Theorem 2.2.1;3. We let N denote the operator $N : f \mapsto F$.

Theorem 2.1.1;3 requires that u vanish for $r \geq 1$. In the problem (1a,b), any solution belongs to $H_0^1(S)$. This fact will allow us to remove the condition that u vanish for $r \geq 1$.

We recall some notations connected with Theorem 2.2.1;3. Let $s_j = j\alpha + 1$. Thus, $(s_j - 1)/\alpha = j$, for $j = 1, 2, \cdots$, so the s_j are the exceptional values that occur in the singularity expansion of $-\Delta$. Set $s_0 = 1$. For a number s with $(s-1)/a \neq$ integer, let

$$J(s) = \max\{j : j\alpha < s - 1\}.$$

The presence of singularities in the solution of (2a,b) is governed by certain linear functionals of the data. Since the boundary data in (2b) has been taken to be 0, one needs only the linear functionals $\Lambda_j''(F)$ defined in (2.2.1;17b). The linear functional $\Lambda_j''(F)$ is well-defined and provided $F \in H^{s-2}(S)$ with $s > s_j$. The following lemma will prove convenient in the analysis of (1).

Lemma 1. *Let s and t satisfy $1 < t < s$. Suppose $s_k < s < s_{k+1}$, $s_j < t < s_{j+1}$ for some integers $j \geq 0$ and $k \geq j$. Let $f \in H^{s-2}(S)$, and let u be the solution of (1a,b). Suppose u vanishes outside S_1. If $j < k$, then $u \in H^{s'}(S)$ for any $s' < s_{j+1}$. If $j = k$, then $u \in H^s(S)$.*

Proof. Suppose first that $j < k$. Since $u \in H^t(S)$, $|\nabla u| \in H^{t-1}(S)$ so $-\Delta u = F \in H^{t-1}(S)$. There are now two possibilities. If $t - 1 > s_{j+1} - 2$ then $F \in H^{s'-2}(S)$ for any $s' < s_{j+1}$. If $t - 1 \leq s_{j+1} - 2$, then $t - 1 < s - 2$, so $F \in H^{t-1}(S)$. In either case, if $k \geq 1$, the linear functionals $\Lambda_l''(F)$ are well defined for $l = 1, \cdots, j$. Since $u \in H^t(S)$ with $t > s_j$ one concludes from Theorem 2.2.1;3 that if $j \geq 1$ $\Lambda_1''(F) = \cdots = \Lambda_j''(F) = 0$. Hence, again from Theorem 2.2.1;3,

$$u \in H^{s'}(S) \text{ for } s' < s, \text{ if } t - 1 > s_{j+1} - 2,$$
$$u \in H^{t+1}(S) \text{ if } t - 1 \leq s_{j+1} - 2.$$

In the first case the proof is complete. In the second case we repeat the argument with t replaced by $t+1$, and eventually we conclude that $u \in H^{s'}(S)$ for any $s' < s_{j+1}$.

Now suppose that $j = k$. Thus, $s_k < t < s < s_{k+1}$. We argue as above, but in this case, the regularity in u cannot be increased beyond s, so we obtain $u \in H^s(S)$. ■

We now apply the lemma. Suppose $f \in H^s(S)$ for some large value of s. The solution $u \in H^1(S)$. From Lemma 1 with $t = 1$ we conclude that

$$u \in H^{s'}(S),\ F \in H^{s'-1}(S),\ \text{for } s' < s_1 \tag{3}$$

Suppose for convenience that the sector S is convex. This means that $\omega < \pi$, $\alpha > 1$, and $s_1 = \alpha + 1 > 2$. Also one sees that $s_1 + 1 < s_2$. Using Theorem 2.2.1;3 we write

$$u = \Lambda_1''(F)v_1 + u_1 \quad \text{with } u_1 \in H^{s'}(S) \text{ for } s' < s_1. \tag{4}$$

Inserting this decomposition of u into the formula for F, we obtain

$$F = f - p_1 u_{1,x_1} - p_2 u_{1,x_2} - p_1 \Lambda_1''(F) v_{1,x_1} - p_2 \Lambda_1''(F) v_{1,x_2} - qu. \tag{5}$$

We see that there are two terms in the right side of (5) that are in $H^{s'-1}(S)$, namely the terms involving v_{1,x_1} and v_{1,x_2}. To continue to a higher order expansion, these terms must be removed. For this we pick a function $v_{1,1} \in H_0^1(S)$ such that $-\Delta v_{1,1} = \chi(p_1 v_{1,x_1} + p_2 v_{1,x_2})$, where χ is a smooth function which vanishes for $r > 1$ and which is $= 1$ in a neighborhood of the origin. Then $u_{1,1} = u - \Lambda_1''(F)v_{1,1}$ satisfies

$$-\Delta u_{1,1} = F_{1,1} := f - p_1 u_{1,x_1} - p_2 u_{1,x_2} - (1-\chi)(p_1 v_{1,x_1} + p_1 v_{1,x_1}) - qu \in H^{s'}(S) \text{ for } s' < s_1$$

Thus $F_{1,1}$ has higher regularity and the expansion can be continued using Theorem 1. The function $v_{1,1}$ is a new singular function that arises from the presence of the lower order terms. One has $v_{1,1} \in H^s(S)$ for $s < s_1 + 1$.

Proceeding along the lines sketched above, it would be possible to develop a corner singularity expansion for the solution of (1) plus the associated bounds on u. Instead, an alternate approach is developed, which relys on the assumption that the coefficients in the operator L are constant. We will find a function ζ_j such that $L\zeta_j = 0$ near the origin, $\zeta_j = 0$ on Γ, and $\Lambda_j''(-\Delta\zeta_j) = 1$. Using the function ζ_1 for example, we can define u_1 by the decomposition $u = \Lambda_1''(F)\zeta_1 + u_1$. Then $\Lambda_1''(-\Delta u_1) = 0$, but now $Lu_1 = f - \Lambda_1''(F)L\zeta_1$ has the same regularity as f, so the expansion can be continued without recourse to an extra singular function as $v_{1,1}$.

To find such functions ζ_j, make the substitution $u = e^{(p_1x_1+p_2x_2)/2}z$. Then $Lu = [-\Delta z + (q + \frac{1}{4}(p_1^2 + p_2^2))z]e^{(p_1x_1+p_2x_2)/2}$, so if $Lu = 0$ then z satisfies

$$-\Delta z + (q + \tfrac{1}{4}(p_1^2 + p_2^2))z = 0.$$

Set $k^2 = q + \frac{1}{4}(p_1^2 + p_2^2)$. If z is a function of the form $z(x,y) = R(r)\sin j\alpha\theta$ we obtain

$$Lu = -e^{(p_1x_1+p_2x_2)/2}(M_jR)\sin j\alpha\theta,$$

where the differential operator M_j is defined by

$$M_jR = R'' + r^{-1}R' - (j^2\alpha^2 + k^2r^2)r^{-2}R.$$

Thus, to find functions u with $Lu = 0$, we select functions R with $M_j R = 0$. Noting that $R(r) = I_{j\alpha}(kr)$ or $R(r) = K_{j\alpha}(kr)$ solves this equation, we see that the functions

$$u = e^{(p_1x_1+p_2x_2)/2} I_{j\alpha}(kr)\sin j\alpha\theta, \tag{6a}$$

$$u = e^{(p_1x_1+p_2x_2)/2} K_{j\alpha}(kr)\sin j\alpha\theta, \tag{6b}$$

satisfy $Lu = 0$. Regarding $I_{j\alpha}$ and $K_{j\alpha}$ recall that

$$\begin{aligned} &I_{j\alpha}(t) \sim A_j t^{j\alpha} + O(t^{(j+1)\alpha}),\ K_{j\alpha}(t) \sim B_j t^{-j\alpha} + O(t^{(-j+1)\alpha}), \quad \text{for } t \text{ small},\\ &I_{j\alpha}(t) \sim t^{-1}e^{t},\ K_{j\alpha}(t) \sim t^{-1}e^{-t}, \quad \text{for } t \text{ large}. \end{aligned} \tag{7}$$

The numbers A_j, B_j are given in [AS, (9.6.7), (9.6.9)].

To define our singular function ζ_j, let ψ_j be a smooth function such that $\psi_j = 0$ in $(0,1)$ and $\psi_j(r) = 0$ outside $(0,2)$. An additional condition on ψ_j is given below. Let $R_j(r)$ satisfy the two point boundary value problem

$$M_j R_j = \psi_j(r),\ R_j(0) = 0,\ R_j(r) \to 0 \text{ as } r \to \infty.$$

Let

$$\zeta_j(x,y) = e^{(p_1x_1+p_2x_2)/2} R_j(r)\sin j\alpha\theta \tag{8}$$

so

$$L\zeta_j = -\tfrac{1}{4} e^{(p_1x_1+p_2x_2)/2} \psi(r)\sin j\alpha\theta. \tag{9}$$

Since $\psi = 0$ near $r = 0$, $R_j(r) \sim I_{j\alpha}(kr)$ so

$$\zeta_j \sim r^{j\alpha} \quad \text{near } r = 0. \tag{10a}$$

Since $\psi = 0$ near $r = \infty$, $R_j(r) \sim K_{j\alpha}(kr)$ so

$$\zeta_j \sim r^{-1}\exp\left\{-\Big([q + \tfrac{1}{4}(p_1^2+p_2^2)r]^{1/2} - \tfrac{1}{2}(p_1x_1+p_2x_2)\Big)\right\} \quad \text{near } r = \infty. \tag{10b}$$

Note that the function $r^\beta \in H^s(S_1)$ if and only if $s < \beta + 1$. Hence, using (10a) and (10b), $\zeta_j \in H^s(S)$ if and only if $s < j\alpha + 1 = s_j$, which conforms with the assertion of Theorem 2.2.1;3 regarding v_j. Since $-\Delta\zeta_j = L\zeta_j - p_1\zeta_{j,x_1} - p_2\zeta_{j,x_2} - q\zeta_j$ and $L\zeta_j$ is smooth, $-\Delta\zeta_j \in H^s(S)$ for $s < s_j - 1$. Hence $\Lambda_j''(-\Delta\zeta_j)$ is well-defined. To calculate $\Lambda_j''(-\Delta\zeta_j)$, we use the formula (2.2.1;17b). This formula requires the Taylor polynomial of degree $\lfloor j\alpha\rfloor - 2$ of $-\Delta\zeta_j$. Since $L\zeta_j \equiv 0$ near $r = 0$, its Taylor polynomial is zero. From (10a) it is seen that the Taylor polynomials of ζ_j, ζ_{j,x_1}, and ζ_{j,x_2} are zero. Hence the Taylor polynomial of $-\Delta\zeta_j$ is zero and we have

$$\Lambda_j''(-\Delta\zeta_j) = \frac{1}{j\pi}\int\int_S (-\Delta\zeta_j) r^{-j\alpha}\sin j\alpha\theta dx. \tag{11}$$

We now assert that

$$\Lambda_j''(-\Delta\zeta_j) = \frac{-1}{j\pi} C_j \int\int_S (L\zeta_j) e^{(-p_1x_1-p_2x_2)/2} K_{j\alpha}(kr)\sin j\alpha\theta dx, \tag{12}$$

where $C_j = k^{j\alpha} B_j^{-1}$ and B_j is the constant occuring in (7). In the proof of (12) we use the adjoint operator L^*, defined by

$$L^* u := -\Delta u - p_1 u_{x_1} - p_2 u_{x_2} + qu.$$

To show (12) write $\Lambda_j(-\Delta\zeta_j) = \Lambda_j(L\zeta_j - p_1\zeta_{j,x_1} - p_2\zeta_{j,x_2} - q\zeta_j) = \Lambda_j(L\zeta_j) - \Lambda_j(p_1\zeta_{j,x_1} + p_2\zeta_{j,x_2} + q\zeta_j)$. Also, define $\tilde{K}(x,y) = e^{(-p_1x_1 - p_2x_2)/2} K_{j\alpha}(kr)$. We must show that

$$\begin{aligned}\int\!\!\int_S L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}\sin j\alpha\theta\}dx \\ = \int\!\!\int_S r^{-j\alpha}\sin j\alpha\theta(p_1\zeta_{j,x_1} + p_2\zeta_{j,x_2} + q\zeta_j)dx.\end{aligned} \tag{13}$$

For this we write

$$\begin{aligned}&\int\!\!\int_S L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}\sin j\alpha\theta\}dx \\ &= \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}\sin j\alpha\theta\}dx \\ &= \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c} \zeta_j\{L^*(r^{-j\alpha}\sin j\alpha\theta) - C_jL^*(\tilde{K}\sin j\alpha\theta)\})dx \\ &\quad - \lim_{\delta\to 0}\int_{r=\delta}\zeta_{j,r}\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}\sin j\alpha\theta\}\delta d\theta \\ &\quad + \lim_{\delta\to 0}\int_{r=\delta}\zeta_j\{-j\alpha r^{-j\alpha-1}\sin j\alpha\theta - C_j\tilde{K}_r\sin j\alpha\theta\}\delta d\theta \\ &\quad - \lim_{\delta\to 0}\int_{r=\delta}(p_1\cos\theta + p_2\sin\theta)\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}\sin j\alpha\theta\}\delta d\theta.\end{aligned}$$

As in the calculation that leads to (6a,b), one sees that $L^*(\tilde{K}\sin j\alpha\theta\}) = 0$. Also, using (7) we see that the integrals over $r = \delta$ tend to zero as $\delta \to 0$. We therefore get

$$\begin{aligned}&\int\!\!\int_S L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}\sin j\alpha\theta\}dx \\ &= \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c}\zeta_j L^*(r^{-j\alpha}\sin j\alpha\theta)dx.\end{aligned}$$

Since $r^{-j\alpha}\sin j\alpha\theta$ is a harmonic function $L^*(r^{-j\alpha}\sin j\alpha\theta) = -p_1(r^{-j\alpha}\sin j\alpha\theta)_{x_1} - p_2(r^{-j\alpha}\sin j\alpha\theta)_{x_2} + qr^{-j\alpha}\sin j\alpha\theta$. Hence

$$\begin{aligned}&\int\!\!\int_S L\zeta_j\{r^{-j\alpha}\sin j\alpha\theta - C_j\tilde{K}\sin j\alpha\theta\}dx \\ &= \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c}\zeta_j\{-p_1(r^{-j\alpha}\sin j\alpha\theta)_{x_1} - p_2(r^{-j\alpha}\sin j\alpha\theta)_{x_2} + qr^{-j\alpha}\sin j\alpha\theta)\}dx \\ &= \lim_{\delta\to 0}\int\!\!\int_{S_\delta^c}(p_1\zeta_{j,x_1} + p_2\zeta_{j,x_2} + q\zeta_j)(r^{-j\alpha}\sin j\alpha\theta)dx \\ &\quad + \lim_{\delta\to 0}\int_{r=\delta}(p_1n_1 + p_2n_2)\zeta_j r^{-j\alpha}\sin j\alpha\theta)\delta d\theta.\end{aligned}$$

Again it is seen that the integral over $r = \delta$ tends to zero as $\delta \to 0$. This completes the proof of (13), and hence of (12).

Using (12) we have

$$\Lambda_j(-\Delta\zeta_j) = \frac{1}{4j\pi}C_j \int\int_S \psi_j(r)K_{j\alpha}(kr)\sin^2 j\alpha\theta dx. \tag{14}$$

The function ψ_j is chosen so that the integral in (14) is non-zero, and is normalized so that

$$\int\int_S \psi_j(r)K_{j\alpha}(kr)\sin^2 j\alpha\theta dx = 4j\pi C_j^{-1}. \tag{15}$$

Then $\Lambda_j(-\Delta\zeta_j) = 1$.

Theorem 1. *There are linear functionals $\Lambda_{L,j}$ and functions ζ_j, $j = 1, 2, \cdots$, with the following properties. (i) $\Lambda_{L,j}$ is a bounded linear functional on $H^{s-2}(S)$ for $s > s_j$, but not for $s \leq s_j$. (ii) $\zeta_j \in H^s(S) \cap H_0^1(S)$ for $s < s_j$ but not for $s \geq s_j$. Also ζ_j is smooth everywhere except at the origin, $L\zeta_j = 0$ in a neighborhood of the origin, and $L\zeta_j = 0$ for large r. (iii) If $s_j < s < s_{j+1}$, $f \in H^{s-2}(S)$ and $u = Af$, then*

$$u_j := u - \sum_{l=1}^{j} \Lambda_{L,l}(f)\zeta_l \in H^s(S), \quad \textit{with } \|u_j\|_s \leq C\|f\|_{s-2}. \tag{16}$$

In case $s_0 < s < s_1$, $u \in H^s(S)$.

Proof. The proof is given in a series of steps.

(a). Let χ be a smooth function which is $\equiv 1$ in a neighborhood of the origin and $\equiv 0$ for $r \geq 1$. Since $u \in H^1(S)$, $\chi u \in H^1(S)$ and $L(\chi u) \in H^{\min(s-2,0)}(S)$. Hence $NL(\chi u) \in H^{\min(s-2,0)}(S)$. Lemma 1 implies that $\chi u \in H^s(S)$. Since $(1-\chi)u$ solves an elliptic problem on a smooth domain, it can be shown that $(1-\chi)u \in H^s(S)$. Hence $u \in H^s(S)$. This gives the decomposition (16) in the case $j = 0$.

(b). Suppose that $s_1 < s < s_2$. From (a), $u \in H^{s'}(S)$ for any $s' < s_1$. Hence $\chi u \in H^{s'}(S)$. Hence $F = NL(\chi u) \in H^{s'-2}(S)$ for any $s' < s_1+1$. Pick a number $s' > s_1$ with $s' < \min(s, s_1+1, s_2)$. Then $\Lambda_1(F)$ is well-defined. We define $\Lambda_{L,1}(f) = \Lambda_1(F)$. Let $u_1^* = \chi u - \Lambda_{L,1}(f)\zeta_1$. Then $\Lambda_{L,1}(Lu_1^*) = \Lambda_1(-\Delta u_1^*) = 0$ so from Theorem 2.1.1;3, $u_1^* \in H^{s'}(S)$. Also $Lu_1^* \in H^{s-2}(S)$. Applying Lemma 1, $u_1^* \in H^s(S)$. Since $(1-\chi)u$ solves an elliptic problem on a smooth domain, it can be shown that $(1-\chi)u \in H^s(S)$. Setting $u_1 = u_1^* + (1-\chi)u$, we have $u = \Lambda_{L,1}(f)\zeta_1 + u_1$ with $u_1 \in H^s(S)$. This gives the decomposition (16) in the case $j = 1$.

(c) We use an argument by induction and a recursive definition of the linear functionals. Suppose the linear functionals $\Lambda_{L,i}$ have been defined for $i < j$. Let $s_j < s < s_{j+1}$, let $f \in H^{s-2}(S)$, and let u be the solution of (1a,b). By induction, we may write $u = \sum_{l=1}^{j-1} \Lambda_{L,l}(f)\zeta_l + u_{j-1}$ with $u_{j-1} \in H^{s'}(S)$ for any $s' < s_{j-1}$. Hence $\chi u_{j-1} \in H^{s'}(S)$. Hence $F = NL(\chi u_{j-1}) \in H^{s'-2}(S)$ for any $s' < s_{j-1}+1$. Pick a number $s' > s_{j-1}$ with $s' < \min(s, s_{j-1}+1, s_2)$. Then $\Lambda_j(F)$ is well-defined. We define $\Lambda_{L,j}(f) = \Lambda_j(F)$. Let $u_j^* = \chi u_{j-1} - \Lambda_{L,j}(f)\zeta_j$. Then $\Lambda_{L,j}(Lu_j^*) = \Lambda_j(-\Delta u_j^*) = 0$ so from Theorem 2.1.1;3, $u_j^* \in H^{s'}(S)$. Also $Lu_j^* \in H^{s-2}(S)$. Applying Lemma 1, $u_j^* \in H^s(S)$. Since $(1-\chi)u_{j-1}$ solves an elliptic problem on a smooth domain, it can be shown that $(1-\chi)u_{j-1} \in H^s(S)$. Setting $u_j = u_j^* + (1-\chi)u_{j-1}$, we have $u = \Lambda_{L,j}(f)\zeta_j + u_j$ with $u_j \in H^s(S)$. This gives the decomposition (16) in the general case. ∎

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It is of interest to give a formula for the constant $K(s,\varepsilon)$ and the norm of the linear functional $\|\tilde{\Lambda}_j\|$. This is done in the following lemma.

Lemma 2. *Let* $u \in H^{s+2}(S) \cap H_0^1(S)$ *and suppose that* $u = 0$ *in* S_1^c. *Then*

$$\|u\|_{s+2} \le K(s,\varepsilon)\|Lu\|_s \tag{15}$$

where $K(s,\varepsilon) = C\varepsilon^{-s-3/2}$. *Also*

$$|\tilde{\Lambda}_j(f)| \le C\varepsilon^{-s_j-3/2+\delta}\|f\|_{s_j+\delta}. \tag{16}$$

Proof. Let $f = Lu$, $F = -\Delta u = \varepsilon^{-1}(f - p_x - qu)$. Since $u \in H^{s+2}(S)$, $\Lambda_l(F) = -$ for $l = 1, 2, \cdot, * * **$, so we have from Lemma 1,

$$\|u\|_{s+2} \le \varepsilon^{-1}(\|f\|_s + C_1\|u\|_{s+1}).$$

Since $s+1 = \theta(s+2) + (1-\theta)$ with $\theta = s/(s+1)$ an interpolation inequality gives

$$\|u\|_{s+1} \le C_2\|u\|_{s+2}^{\theta}\|u\|_1^{1-\theta}.$$

Note the inequality $a^\theta b^{1-\theta} \le a + b$. Hence

$$ab = (\delta^{1/\theta}a)^\theta \cdot (\delta^{-1/(1-\theta)}b)^{1-\theta} \le \delta^{1/\theta}a + \delta^{-1/(1-\theta)}b.$$

Applying this with $a = \|u\|_{s+2}$, $b = \|u\|_1$, we obtain

$$\|u\|_{s+2}^{\theta}\|u\|_1^{1-\theta} \le \delta^{1/\theta}\|u\|_{s+2} + \delta^{-1/(1-\theta)}\|u\|_1$$

so

$$\|u\|_{s+1} \le \delta^{1/\theta}C_2\|u\|_{s+2} + \delta^{-1/(1-\theta)}C_2\|u\|_1$$

so

$$\|u\|_{s+2} \le \varepsilon^{-1}\|f\|_s + \delta^{1/\theta}\varepsilon^{-1}C_1C_2\|u\|_{s+2} + \delta^{-1/(1-\theta)}\varepsilon^{-1}C_1C_2\|u\|_1.$$

Choosing δ so that $\delta^{1/\theta}\varepsilon^{-1}C_1C_2 = \frac{1}{2}$ we obtain

$$\|u\|_{s+2} \le 2\varepsilon^{-1}\|f\|_s + C_3\varepsilon^{-1-s}\|u\|_1.$$

From an energy inequality and the fact that $u = 0$ outside S_1 one obtains $\|u\|_1 \le C_4\varepsilon^{-1/2}\|f\|_0$. We therefore have

$$\|u\|_{s+2} \le 2\varepsilon^{-1}\|f\|_s + C_5\varepsilon^{-3/2-s}\|f\|_0.$$

which proves (15).

To show (16) we write

$$\begin{aligned} |\tilde{\Lambda}_j(f)| &= |\Lambda_j(F_j)| \\ &\le C\varepsilon^{-1}[\|f_j\|_{s_j+\delta} + \|u_j\|_{s_j+1-\delta}] \\ &\le C_1\varepsilon^{-1}[\|f\|_{s_j+\delta} + \varepsilon^{-s_j-1/2+\delta}\|f\|_{s_j+\delta}]. && \text{[using (15)]} \end{aligned}$$

Next we consider the problem

$$Nu := -\Delta u + qu = f \text{ in } S,\ u = 0 \text{ on } \Gamma. \tag{17}$$

In this case, $\varepsilon = 1$, $k^2 = 4q$,

$$M_j R = r^2 R'' + rR' + (j^2\alpha^2 + 4qr^2)R,$$

and $\zeta_j^* = R_j(r)\sin j\alpha\theta$. We ask: can the condition of Theorem 2, that $u \equiv 0$ outside S_1^c, be removed? To investigate this, consider the transformation of independent variable

$$\bar{x} = ax,\ \bar{y} = ay,\ \bar{r} = ar,\ \bar{u}(\bar{x}, \bar{y}) = u(x, y),\ \bar{f}(\bar{x}, \bar{y}) = f(x, y).$$

so $\Delta u = a^2 \bar{\Delta}\bar{u}$, so (15) implies that

$$\bar{N}\bar{u} := -\bar{\Delta}\bar{u} + a^{-2}q\bar{u} = a^{-2}\bar{f},\ \bar{u} = 0 \text{ on } \Gamma. \tag{18}$$

Suppose $u \equiv 0$ for $r \geq 1$, so $\bar{u} \equiv 0$ for $\bar{r} \geq a$. Let ζ_j^* be a non-normalized singular function for the problem (17). Let $\bar{\psi}^{(}\bar{r}) = \psi^{(}r)$, $\bar{\zeta}_j^*(\bar{r}) = \zeta_j^*(r)$. It is then seen from (11) that

$$\bar{r}^2 \bar{R}_j'' + \bar{r}\bar{R}_j' + (j^2\alpha^2 + 4qa^{-2}\bar{r}^2)\bar{R} = \bar{\psi},\ \bar{R}_j(0) = 0,\ \bar{R}_j(\rho) \to 0 \text{ as } \bar{r} \to \infty.$$

Therefore the function $\bar{\zeta}_j^*$ satisfies

$$-\bar{\Delta}\bar{\zeta}_j^* + a^{-2}q\bar{\zeta}_j^* = -\bar{\psi}(\bar{r})\sin j\alpha\theta.$$

The decomposition

$$u = \sum_{l=1}^{j} a_j\zeta_j + u_j,\ \|u_j\|_{s+2,S_b} \leq K(b)\|f\|_s, \tag{19}$$

then becomes

$$\bar{u} = \sum_{l=1}^{j} a_j\bar{\zeta}_j + \bar{u}_j. \tag{20}$$

To estimate the remainder $\bar{u}_j$ we note the formulas

$$\int\int_S |D^{s+2}u_j|^2 dxdy = a^{2(s+2)}a^{-2}\int\int_S |\bar{D}^{s+2}\bar{u}_j|^2 d\bar{x}d\bar{y},$$

$$\int\int_S |D^s f|^2 dxdy = a^{2s}a^{-2}\int\int_S |\bar{D}^s\bar{f}|^2 d\bar{x}d\bar{y}.$$

Hence the estimate for u_j in (19) becomes

$$\begin{aligned}\int\int_S |\bar{D}^{s+2}\bar{u}_j|^2 d\bar{x}d\bar{y} &= a^{-2s-4}a^2a^{2s}a^{-2}\int\int_S |\bar{D}^s\bar{f}|^2 d\bar{x}d\bar{y}\\ &= a^{-4}\int\int_S |\bar{D}^s\bar{f}|^2 d\bar{x}d\bar{y}\\ &= \int\int_S |\bar{D}^s(a^{-2}\bar{f})|^2 d\bar{x}d\bar{y}.\end{aligned}$$

Thisis the appropriate estimate for the decomposition (20).