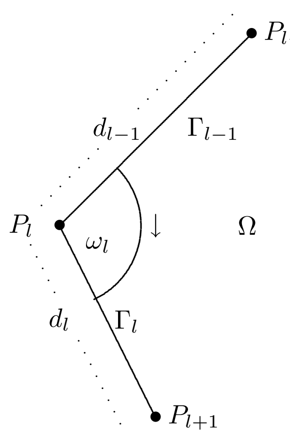
3 Decomposition of the solution in a polygon

The equation $-\Delta u = f$ is considered in a polygonal domain Ω with either a Dirichlet condition or a Neumann condition imposed on each side of the boundary. An expansion of the solution into a sum of singular functions plus a smooth remainder is given. As a consequence, the solution has a certain smoothness if and only if certain linear functionals vanish. A priori inequalities for the remainder and the coefficients of the expansion are given. The result is a consequence of the results given for boundary value problems on a sector in §§II.2.2, II.2.3, and II.2.4.

Let Ω be a plane domain with polygonal boundary Γ. We require some notation. Let the sides of Ω be $\Gamma_1, \cdots, \Gamma_M$, and let the vertices be $P_1, \cdots P_M$, arranged in a counterclockwise order with P_l and P_{l+1} the two vertices of Γ_l, so Γ_l and Γ_{l-1} are the two sides emanating from P_l. Let $\mathbf{n}_l = [n_{l1}, n_{l2}]$ denote the outward pointing normal vector to side Γ_l, and let ω_l denote the interior angle of Γ at P_l, measured in a counterclockwise direction from Γ_l to Γ_{l-1}. We assume that $0 < \omega_l \leq 2\pi$. Thus, the case of a slit domain ($\omega_l = 2\pi$) is included. Let d_l be the length of Γ_l, and let $d_{\min} = \min_l d_l$, $d_{\max} = \max_l d_l$. These quantities are shown in the figure.



If g is a function on Γ, let g_l denote the restriction of g to the side Γ_l, and write $g = \{g_l\}$. We also consider g_l as being defined on the interval $[0, d_l]$, with $g_l(0) = g_l(P_l)$ and $g_l(d_l) = g_l(P_{l+1})$. We let γ_l denote the trace map associated with the side Γ_l; thus, $\gamma_l u = g_l$ denotes the restriction of u to the side Γ_l. u_n is the outward pointing normal derivative of u. Thus, on the side Γ_l, $u_n = \gamma_l \mathbf{n}_l \cdot \nabla u$. Let (r_l, θ_l) denote polar coordinates centered at the vertex P_l, with Γ_l lying on $\theta_l = 0$ and therefore with Γ_{l-1} lying on $\theta_l = \omega_l$. If $\omega_l = 2\pi$, the domain includes a slit, and P_l is the tip of the slit. If $\omega_l = \pi$, the vertex P_l lies in the interior of a side of Ω, but there is a possible change of boundary condition or a jump in the boundary data at P_l.

Let $\mathcal{M} = \{1, \cdots, M\}$ denote the first M integers, partitioned into two subsets, $\mathcal{M} = \mathcal{M}_D \cup \mathcal{M}_N$, with $\mathcal{M}_D \cap \mathcal{M}_N = \emptyset$. We consider the mixed boundary value problem

$$(1a) \qquad -\Delta u = f \text{ in } \Omega,$$

$$(1b) \qquad u = g_l \text{ on } \Gamma_l, \quad l \in \mathcal{M}_D,$$

$$(1c) \qquad D_{n_l} u = h_l \text{ on } \Gamma_l, \quad l \in \mathcal{M}_N.$$

If $\mathcal{M}_N = \emptyset$, (3.1) is the Dirichlet problem. If $\mathcal{M}_D = \emptyset$, (3.1) is the Neumann problem, and in order to have a solution the data f, h must satisfy the compatibility condition $\int\int_\Omega dx + \int_\Gamma h ds = 0$. We say that a vertex P_l is of *Dirichlet type* if $l, l-1 \in \mathcal{M}_D$; P_l is of *Neumann type* if $l, l-1 \in \mathcal{M}_N$; P_l is of *mixed type* if P_l is neither of Dirichlet type or of Neumann type. We shall use the sector analysis given in the 3 preceding sections. Thus, we set $\alpha_l = \pi/\omega_l$, and we define $J_l(s)$ by

$$J_l(s) = \begin{cases} \max\{j : j\alpha_l < s-1\}, & P_l \text{ is of Dirichlet or Neumann type,} \\ \max\{j : (j-\frac{1}{2})\alpha_l < s-1\}, & P_l \text{ is of mixed type,} \end{cases}$$

and we let $\Lambda_j^{(l)}$ denote the linear functionals appropriate to the vertex P_l as defined in one of the preceding sections. We also need the singular functions associated with each vertex, which we write $v_j^{(l)}$. From the preceding sections, these singular functions are given by the following formulas.

P_l of Dirichlet type

$$v_j^{(l)}(x) = \begin{cases} r_l^{j\alpha_l} \sin j\alpha_l\theta_l, & j\alpha_l \neq \text{integer,} \\ [\theta_l \cos j\alpha_l\theta_l + (\ln r_l)\sin j\alpha_l\theta_l] r_l^{j\alpha_l}, & j\alpha_l = \text{integer.} \end{cases}$$

P_l of Neumann type

$$v_j^{(l)}(x) = \begin{cases} r_l^{j\alpha_l} \cos j\alpha_l\theta_l, & j\alpha_l \neq \text{integer,} \\ [\theta_l \sin j\alpha_l\theta_l - (\ln r_l)\cos j\alpha_l\theta_l] r_l^{j\alpha_l}, & j\alpha_l = \text{integer.} \end{cases}$$

P_l of mixed type with $l \in \mathcal{M}_D$, $l-1 \in \mathcal{M}_N$

$$v_j^{(l)}(x) = \begin{cases} r_l^{(j-\frac{1}{2})\alpha_l} \sin(j-\frac{1}{2})\alpha_l\theta_l, & (j-\frac{1}{2})\alpha_l \neq \text{integer,} \\ [\theta_l \cos(j-\frac{1}{2})\alpha_l\theta_l + (\ln r_l)\sin(j-\frac{1}{2})\alpha_l\theta_l] r_l^{(j-\frac{1}{2})\alpha_l}, & (j-\frac{1}{2})\alpha_l = \text{integer.} \end{cases}$$

P_l of mixed type with $l \in \mathcal{M}_N$, $l-1 \in \mathcal{M}_D$

$$v_j^{(l)}(x) = \begin{cases} r_l^{(j-\frac{1}{2})\alpha_l} \cos(j-\frac{1}{2})\alpha_l\theta_l, & (j-\frac{1}{2})\alpha_l \neq \text{integer,} \\ [\theta_l \sin(j-\frac{1}{2})\alpha_l\theta_l + (\ln r_l)\cos(j-\frac{1}{2})\alpha_l\theta_l] r_l^{(j-\frac{1}{2})\alpha_l}, & (j-\frac{1}{2})\alpha_l = \text{integer.} \end{cases}$$

We need a data space for the problem (1). For $s \geq 2$ we define $\mathcal{Y}^s(\Omega)$ to be the collection of triples $\{f, g, h\}$ where $f \in H^{s-2}(\Omega)$ and where

$$g = \{g_l \in H^{s-1/2}(\Gamma_l) : l \in \mathcal{M}_D\},\ h = \{h_l \in H^{s-3/2}(\Gamma_l) : l \in \mathcal{M}_N\},$$

and such that for each l such that P_l is a vertex of Dirichlet type, $g_l(0) = g_{l-1}(d_{l-1})$. We require a partition of unity associated with the polygon Γ. Let U_l and V_l, $l = 1, \cdots, M$, be open sets with $P_l \in U_l \subset\subset V_l$. Suppose also that $\Gamma \subset \cup V_l$ and that $V_l \cap V_m = \emptyset$ unless $m = l \pm 1 \bmod M$. Finally, suppose that $U_l \cap V_m = \emptyset$ for each $l \neq m$. Let $\chi_l \in C_0^\infty(V_l)$ with $\chi_l \equiv 1$ on and such that $\sum_1^M \chi_l \equiv 1$ on Γ. Thus, $\chi_1, \cdots, \chi_M$ is a partition of unity on Γ. We set $\Gamma_0 = 1 - \sum_1^M \chi_l$, so that $\chi_0 \in C_0^\infty(\Omega)$.

Theorem 1. *Let $s \geq 2$. Suppose that $(s-1)/\alpha \neq$ integer in the case that there is at least one vertex of either Dirichlet or Neumann type, and $2(s-1)/\alpha \neq$ odd integer in the case that there is at least one vertex of mixed type. Let $\{f, g, h\} \in \mathcal{Y}^s(\Omega)$, and let $u \in H^1(\Omega)$ be a solution of (1) (u is unique unless $\mathcal{M}_D = \emptyset$). Then*

$$u = \sum_{l=1}^{M} \sum_{j=1}^{J_l(s)} \hat{\Lambda}_j^{(l)}\{f, g, h\} v_j^{(l)} + W \tag{2}$$

where $W \in H^s(\Omega)$ and satisfies the inequality

$$||W||_{H^s(\Omega)} \le C||\{f,g,h\}||_{X^s}.$$

Proof. The proof is divided into a series of steps.

(i) From regularity theory for an elliptic equation in a smooth domain, we know that if $\Omega' \subset\subset \Omega$ and if $P_l \notin \overline{\Omega'}$, $l = 1, \cdots, M$, then $u \in H^s(\Omega')$ and $||u||_{s,\Omega'} \le C||\{f,g,h\}||_{X^s}$. The constant C depends on the regions Ω and Ω'.

(ii) Let $u_l = \chi_l u$. Then

$$-\Delta u_l = f_l \equiv \chi_l f - u\Delta\chi_l - 2\nabla\chi_l\nabla u \text{ in } \Omega, \tag{3a}$$

Since $u_0 = 0$ near Γ, (i) implies that $u_0 \in H^s(\Omega)$ and

$$||u_0||_{s,\Omega} \le C||f_0||_{s-2,\Omega} \le ||\{f,g,h\}||_{X^s}. \tag{3}$$

(iii) Let S_l denote the sector with apex at P_l and whose sides are the rays obtained by extending Γ_l and Γ_{l-1}. The function u_l vanishes outside $\Omega \setminus S_l$, and without loss of generality it may be assumed that $u_l = 0$ for $r_l \ge 1$. If $l \in \mathcal{M}_D$, then $u_l = \chi_l g_l$ on Γ_l. We have

$$\begin{aligned} u_l = g_0^{(l)} &:= \chi_l g_l \text{ on } \Gamma_l, \text{ if } l \in \mathcal{M}_D, \\ D_{n_l} u_l = h_0^{(l)} &:= \chi_l h_l + u D_{n_l}\chi_l \text{ on } \Gamma_l, \text{ if } l \in \mathcal{M}_N, \end{aligned} \tag{3b}$$

$$\begin{aligned} u_l = g_1^{(l)} &:= \chi_l g_{l-1} \text{ on } \Gamma_{l-1}, \text{ if } l-1 \in \mathcal{M}_D, \\ D_{n_{l-1}} u_{l-1} = h_0^{(l)} &:= \chi_l h_{l-1} + u D_{n_{l-1}}\chi_l \text{ on } \Gamma_{l-1}, \text{ if } l-1 \in \mathcal{M}_N. \end{aligned} \tag{3c}$$

Thus, u_l satisfies the problem (3a,b,c) in S_l. Evidently, if $l \in \mathcal{M}_D$, then $g_0^{(l)} \in H^{s-1/2}(\Gamma_l)$, and similarly, if $l-1 \in \mathcal{M}_D$, then $g_1^{(l)} \in H^{s-1/2}(\Gamma_{l-1})$. If P_l is a vertex of Dirichlet type, then $g_0^{(l)}(0) = g_l(0) = g_{l-1}(d_{l-1}) = g_1^{(l)}(0)$, so the compatibility condition is satisfied and $\{f_l.g_0^{(l)}, g_1^{(l)}\} \in \mathcal{Y}_D^s(S_l)$. If $l \in \mathcal{M}_N$, we note that $D_{n_l}\chi_l$ vanishes near P_l, so $h_0^{(l)} = h_l$ near P_l. Far from P_l, $u_l|_{\Gamma_l} \in H^{s-1/2}_{\text{loc}}(\Gamma_l)$. Hence elliptic regularity implies that $h_0^{(l)} \in H^{s-3/2}(\Gamma_l)$. A similar reasoning applies if $l-1 \in \mathcal{M}_N$. Hence, if P_l is of Neumann type or mixed type, the data belongs to the appropriate data space. Suppose, for example, that P_l is a vertex of mixed type, with $l \in \mathcal{M}_D$, $l-1 \in \mathcal{M}_N$. Applying Theorem II.2.4;2, we write

$$u_l(x) = \sum_{j=1}^{J_l(s)} \Lambda_j^{(l)}\{f_l, g_0^{(l)}, h_1^{(l)}\} v_j^{(l)}(x) + W_l(x, s-1)$$

with $W_l \in H^s(\Omega)$. We now define new functionals $\hat{\Lambda}_j^{(l)}$ on $X^s(\Omega)$ by

$$\hat{\Lambda}_j^{(l)}\{f,g,h\} = \Lambda_j^{(l)}\{f_l, g_0^{(l)}, h_1^{(l)}\}. \tag{4}$$

Assembling these results, we obtain (2). ■