2.4 The equation $-\Delta u = f$ in a sector - mixed boundary conditions

This section treats the mixed problem

$$-\Delta u = f \text{ in } S, \tag{1a}$$

$$u(x,0) = g_0(x),\ x > 0 \tag{1b}$$

$$\begin{aligned} D_\nu u(r\cos\omega, r\sin\omega) = &- (\sin\omega)u_x(r\cos\omega, r\sin\omega) \\ &+ (\cos\omega)u_y(r\cos\omega, r\sin\omega) := h_1(r),\ r > 0. \end{aligned} \tag{1c}$$

in a sector with angle ω. The data $\{f, g_0, h_1\}$ is assumed to have a certain amount of regularity. There is obtained an expansion of a weak solution of (1) into a set of singular functions plus a smooth remainder.

We shall use the data space $\mathcal{Y}^s_M(S)$ for the problem (1) defined in §II.1. That is, for $s \geq 2$, $\mathcal{Y}^s_M(S)$ is the set of all triples $\{f, g_0, h_1\} \in H^{s-2}(S) \times H^{s-1/2}(R_+) \times H^{s-3/2}(R_+)$. A function $u \in H^1(S)$ is defined to be a weak solution of (1) if $u = g_0$ on Γ_0 and if u satisfies

$$\int\int_S \nabla u \cdot \nabla\varphi dx = \int_0^\infty h_1(r)\varphi(r\cos\omega, r\sin\omega)dr + \int\int_S f\varphi dx \text{ for } \varphi \in H^1(S) \text{ with } \varphi = 0 \text{ on } \Gamma_0.$$

We shall consider weak solutions which also satisfy

$$u(x) \equiv 0 \text{ for } r = \sqrt{x_1^2 + x_2^2} > 1.$$

If $\omega \neq \pi/2$ or $\omega \neq 3\pi/2$, for any pair $\{g_0, h_1\} \in H^{s-1/2}(R_+) \times H^{s-3/2}(R_+)$, there is a function $H \in H^1(S)$ such that $H = g_0$ on Γ_0 and $D_\nu H = h_1$ on Γ_1. Thus, there is no issue of incompatibility of the boundary data at 0 in these cases. If $\omega = \pi/2$, in order for there to be a function $H \in H^s(S)$ such that $H(x,0) = g_0(x)$ for $x > 0$ and $-H_x(0,y) = h_1(y)$ for $y > 0$, the boundary data $\{g_0, h_1\} \in H^{s-1/2}(R_+) \times H^{s-3/2}(R_+)$ must satisfy the compatibility condition

$$\int_0^\infty x^{-1}|g_0'(x) + h_1(x)|^2 dx < \infty \text{ if } s = 2,$$

$$g_0'(0) = -h_1(0) \text{ if } s > 2.$$

Similar conditions hold in the case $\omega = 3\pi/2$. In this case, in order for there to be a function $H \in H^s(S)$ such that $H(x,0) = g_0(x)$ for $x > 0$ and $H_x(0,y) = h_1(y)$ for $y < 0$, the boundary data $\{g_0, h_1\} \in H^{s-1/2}(R_+) \times H^{s-3/2}(R_+)$ must satisfy the compatibility condition

$$\int_0^\infty x^{-1}|g_0'(x) - h_1(x)|^2 dx < \infty \text{ if } s = 2,$$

$$g_0'(0) = h_1(0) \text{ if } s > 2.$$

As will be seen, the singular expansions will introduce singular functions to remove these incompatibilities in the boundary data at the origin in the special cases $\omega = \pi/2$ and $\omega = 3\pi/2$.

Since $u \in H^s(S)$ for $0 < s < 1$, $\hat{u}(\zeta, \theta)$ is analytic in the half plane $\eta < 0$. From Lemma II.1;5, $\hat{u} = \mathcal{K}u$ satisfies the two point boundary value problem

$$-\hat{u}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{u}(\zeta,\theta) = \hat{f}(\zeta - 2i, \theta),\ \eta < 0, \tag{2a}$$

$$\hat{u}(\zeta, 0) = \hat{g}_0(\zeta - i),\ \hat{u}(\zeta,\omega) = \hat{h}_1(\zeta - i),\quad \eta < 0. \tag{2b}$$

The problem (2) can be solved exactly. The solution is given by the formula

$$\begin{aligned} \hat{u}(\zeta,\theta) = &\int_0^\theta \hat{f}(\zeta - 2i, \varphi)\frac{\cosh\zeta(\omega-\theta)\sinh\zeta\varphi}{\zeta\cosh\zeta\omega}d\varphi \\ &+ \int_\theta^\omega \hat{f}(\zeta - 2i, \varphi)\frac{\sinh\zeta\theta\cosh\zeta(\omega-\varphi)}{\zeta\cosh\zeta\omega}d\varphi \\ &+ \hat{g}_0(\zeta)\frac{\cosh\zeta(\omega-\theta)}{\cosh\zeta\omega} + \hat{h}_1(\zeta - i)\frac{\sinh\zeta\theta}{\zeta\cosh\zeta\omega}. \end{aligned} \tag{3}$$

Now suppose $\{f, g_0, h_1\} \in \mathcal{Y}^s_M(S)$ with $s > 2$. Suppose $s \neq$ integer, and write $s = n + \sigma$ with $0 < \sigma < 1$. Note that $s - \frac{1}{2} = n + \sigma - \frac{1}{2}$, and $-\frac{1}{2} < \sigma - \frac{1}{2} < \frac{1}{2}$. Hence n is the closest integer to $s - \frac{1}{2}$. Since $g_0 \in H^{s-1/2}(R_+)$, from Theorem I.2;6, $\hat{g}_0(\zeta)$ has a meromorphic extension to the half plane $\eta < s - 1$ given by

$$\hat{g}_0(\zeta) = \hat{T}_{0,n-1}(\zeta) + \hat{R}_{0,n-1}(\zeta),$$

where

$$\hat{T}_{0,n-1}(\zeta) = \frac{-i}{\sqrt{2\pi}} \sum_{k=0}^{n-1} \frac{g_0^{(k)}(0)}{k!} \frac{1}{\zeta - ki},$$

and where $\hat{R}_{0,n-1}(\zeta)$ is analytic in $\eta < s - 1$ and is given by

$$\hat{R}_{0,n-1}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty [g_0^*(\tau) - T^*_{0,n-1}(\tau)] e^{-i\zeta\tau} d\tau.$$

Similarly, $n - 1$ is the closest integer to $s - \frac{3}{2}$. Since $h_1 \in H^{s-3/2}(R_+)$, from Theorem I.2;6, $\hat{h}_1(\zeta)$ has a meromorphic extension to the half plane $\eta < s - 2$ given by

$$\hat{h}_1(\zeta) = \hat{T}_{1,n-2}(\zeta) + \hat{R}_{1,n-2}(\zeta),$$

where

$$\hat{T}_{1,n-2}(\zeta) = \frac{-i}{\sqrt{2\pi}} \sum_{k=0}^{n-2} \frac{h_1^{(k)}(0)}{k!} \frac{1}{\zeta - ki},$$

and where $\hat{R}_{1,n-2}(\zeta)$ is analytic in $\eta < s - 2$ and is given by

$$\hat{R}_{1,n-2}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty [h_1^*(\tau) - T^*_{1,n-2}(\tau)] e^{-i\zeta\tau} d\tau.$$

Finally, since $f \in H^{s-2}(S)$, Theorem I.4;6 implies that $\hat{f}(\zeta, \theta)$ has a meromorphic extension to the half plane $\eta < s - 3$ given by

$$\hat{f}(\zeta, \theta) = \hat{T}_{n-3}(\zeta, \theta) + \hat{R}_{n-3}(\zeta, \theta).$$

where

$$\hat{T}_{n-3}(\zeta, \theta) = \frac{-i}{\sqrt{2\pi}} \sum_{0 \leq k+m \leq n-3} \frac{\cos^k \theta \sin^m \theta}{k! m!} [D^k_{x_1} D^m_{x_2} f(0,0)] \frac{1}{\zeta - (k+m)i},$$

and where $\hat{R}_{n-3}(\zeta, \theta)$ is analytic in $\zeta < s - 3$ and is given by

$$\hat{R}_{n-3}(\zeta, \theta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty [f^*(\tau, \theta) - T^*_{l,n-2}(\tau, \theta)] e^{-i\zeta\tau} d\tau.$$

Suppose s is chosen so that $s \neq$ integer and $2(s-1)/\alpha \neq$ odd integer. Then the integrands in (3) are finite on $\eta = s - 1$, and are meromorphic in $\eta < s - 1$, with poles at the points $\zeta_{j,0} = (j - \frac{1}{2})\alpha i$, $j = 1, \cdots, J(s)$ and $\zeta_{0,k} = ki$, $k = 0, \cdots, n-1$. Here we define

$$J(s) = \max\{j : (j - \tfrac{1}{2})\alpha < s - 1\}.$$

For $\eta \leq s - 1$, $\eta \neq$ integer, $2\eta/\alpha \neq$ odd integer, define a function $w^*(\tau, \theta, \eta)$ by

$$\begin{aligned} w^*(\tau, \theta, \eta) = {} & \frac{1}{\sqrt{2\pi}} \int_0^\theta \int_{\xi=-\infty}^\infty \hat{f}(\zeta - 2i, \varphi) \frac{\cosh \zeta(\omega - \theta) \sinh \zeta\varphi}{\zeta \cosh \zeta\omega} e^{i\zeta\tau} d\xi d\varphi \\ & + \frac{1}{\sqrt{2\pi}} \int_\theta^\omega \int_{\xi=-\infty}^\infty \hat{f}(\zeta - 2i, \varphi) \frac{\sinh \zeta\theta \cosh \zeta(\omega - \varphi)}{\zeta \cosh \zeta\omega} e^{i\zeta\tau} d\xi d\varphi \\ & + \frac{1}{\sqrt{2\pi}} \int_{\xi=-\infty}^\infty \hat{g}_0(\zeta) \frac{\cosh \zeta(\omega - \theta)}{\cosh \zeta\omega} e^{i\zeta\tau} d\xi + \frac{1}{\sqrt{2\pi}} \int_{\xi=-\infty}^\infty \hat{h}_1(\zeta - i) \frac{\sinh \zeta\theta}{\zeta \cosh \zeta\omega} e^{i\zeta\tau} d\xi. \end{aligned} \tag{4}$$

where the integral is taken on the line $\Im\zeta = \eta$. The integrals in (4) are well defined by the conditions on s. We define $V^*(\tau, \theta, \eta) = u^*(\tau, \theta) - w^*(\tau, \theta, \eta)$. As usual, we let $V(x, \eta)$ and $w(x, \eta)$ denote the corresponding functions in the x-variable. The following lemma gives some properties of the decomposition $u = V + w$.

Lemma 1. *Suppose* $\{f, g_0, h_1\} \in \mathcal{Y}^s_M(S)$ *with* $s > 2$, $s \neq$ *integer,* $2(s-1)/\alpha \neq$ *odd integer. Let* u *be the solution of (1), and suppose* $u \equiv 0$ *for* $r > 1$*. Then* $u(x) = V(x, s-1) + w(x, s-1)$ *where* w *is given by (4), and where* V *is the sum of the residues of the integrals in (4) arising from the poles of the integrands in the strip* $0 < \Im\zeta < s-1$*. The functions* $V(x, s-1)$ *and* $w(x, s-1)$*, regarded as functions of* (x, ω) *with* $x \in S(\omega)$*, are jointly continuous in these variables. For each* $a > 0$ *and* $j = 0, \cdots, n$, $r^{j-s}D^j w \in H^0(S_a)$ *and* $r^{j-n}D^j w \in H^\sigma(S_a)$*, where* S_a *is the truncated sector of radius* a*. There is a constant* $C(s,a) > 0$ *such that*

$$||r^{j-s}D^j w(\cdot, s-1)||_{0,S_a} \leq C(a)||\{f, g_0, h_1\}||_{\mathcal{Y}^s_M(S)},$$

$$||r^{j-n}D^j w(\cdot, s-1)||_{\sigma,S_a} \leq C(a)||\{f, g_0, h_1\}||_{\mathcal{Y}^s_M(S)}.$$

Also,

$$D^m_x w(0, s-1) = 0 \text{ for } 0 \leq m \leq n-2. \tag{8}$$

Proof. Since $\hat{u}$ solves (2),

$$u^*(\tau, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\zeta, \theta) e^{i\zeta\theta} d\xi = w^*(\tau\theta\eta) \text{ for } \eta < 0.$$

We move the line of integration in (4) from $\Im\zeta = \eta < 0$ to $\Im\zeta = s-1$. Using the Cauchy integral formula and an estimate for some integrals on the lines $\Re\zeta = \pm\Xi$, it is seen that $V^*(\tau, \theta, s-1)$ is the sum of residues of the four integrals in (4) at the poles. The continuity of V and w as a function of ω follows from the fact that the location of the poles and residues of the poles are continuous functions of ω. The proof of the remaining properties of w follows the corresponding portion of the proof of Theorem II.2.2;1. ■

We now calculate the function $V(x, s-1)$. Write

$$V(x, s-1) = \sum_1^{J(s)} v_{j,0}(x) + \sum_0^{[s]-1} v_{0,k}(x),$$

where $v_{j,0}(x)$ is the contribution to V arising from the pole at $\zeta_{j,0}$, and $v_{0,k}(x)$ is the contribution to V arising from the pole at $\zeta_{0,k}$. The calculation is divided into the cases when the poles $\zeta_{j,0}$ and $\zeta_{0,k}$ are distinct, and the confluent case when $\zeta_{j,0} = \zeta_{0,k}$ for some j and k,

Calculation of $v_{j,0}$ *in the case of distinct poles*

To calculate the residues at the points $\zeta_{j,0}$ we use the formula

$$\lim_{\zeta \to \zeta_{j,0}} \frac{\zeta - \zeta_{j,0}}{\cosh \zeta\omega} = (-1)^j \frac{i}{\omega}.$$

We have

$$\begin{aligned}\lim_{\zeta \to \zeta_{j,0}} &(\zeta - \zeta_{j,0}) \hat{f}(\zeta - 2i, \varphi) \frac{\cosh \zeta(\omega - \theta) \sinh \zeta\varphi}{\zeta \cosh \zeta\omega} e^{i\zeta\tau} \\ &= (-1)^j \frac{2i}{(2j-1)\pi} \hat{f}(((j - \tfrac{1}{2})\alpha - 2)i, \varphi) e^{-(j-\frac{1}{2})\alpha\tau} \cos(j - \tfrac{1}{2})\alpha(\omega - \theta) \cos(j - \tfrac{1}{2})\alpha\varphi,\end{aligned}$$

$$\begin{aligned}\lim_{\zeta \to \zeta_{j,0}} &(\zeta - \zeta_{j,0}) \hat{f}(\zeta - 2i, \varphi) \frac{\sinh \zeta\theta \cosh \zeta(\omega - \varphi)}{\zeta \cosh \zeta\omega} e^{i\zeta\tau} \\ &= (-1)^j \frac{2i}{(2j-1)\pi} \hat{f}(((j - \tfrac{1}{2})\alpha - 2)i, \varphi) e^{-(j-\frac{1}{2})\alpha\tau} \sin(j - \tfrac{1}{2})\alpha\theta \cos(j - \tfrac{1}{2})\alpha(\omega - \varphi),\end{aligned}$$

$$\lim_{\zeta \to \zeta_{j,0}} (\zeta - \zeta_{j,0}) \hat{g}_0(\zeta) \frac{\cosh \zeta(\omega - \theta)}{\cosh \zeta\omega} e^{i\zeta\tau} = (-1)^j \frac{i}{\omega} \hat{g}_0((j - \tfrac{1}{2})\alpha i) e^{-(j-\frac{1}{2})\alpha\tau} \cos(j - \tfrac{1}{2})\alpha(\omega - \theta),$$

$$\lim_{\zeta\to\zeta_{j,0}}(\zeta-\zeta_{j,0})\hat{h}_1(\zeta-i)\frac{\sinh\zeta\theta}{\zeta\cosh\zeta\omega}e^{i\zeta\tau}=(-1)^j\frac{2i}{(2j-1)\pi}\hat{h}_1(((j-\tfrac{1}{2})\alpha-1)i)e^{-(j-\frac{1}{2})\alpha\tau}\sin(j-\tfrac{1}{2})\alpha\theta.$$

The function $v^*_{j,0}$ is calculated from the formula

$$v^*_{j,0}=\frac{1}{\sqrt{2\pi}}\cdot 2\pi i\cdot\lim_{\zeta\to\zeta_{j,0}}\;=\sqrt{2\pi}i\cdot\lim_{\zeta\to\zeta_{j,0}}\;.$$

Using this and the fact that $\cos(j-\frac{1}{2})\alpha(\omega-\theta)=(-1)^{j+1}\sin(j-\frac{1}{2})\alpha\theta$, we obtain

$$\begin{aligned}v_{j,0}(\tau,\theta)=&\frac{2}{2j-1}\sqrt{\frac{2}{\pi}}r^{(j-\frac{1}{2})\alpha}\sin(j-\tfrac{1}{2})\alpha\theta\int_0^\omega\hat{f}(((j-\tfrac{1}{2})\alpha-2)i,\varphi)\sin(j-\tfrac{1}{2})\alpha\varphi d\varphi\\&+r^{(j-\frac{1}{2})\alpha}\sin(j-\tfrac{1}{2})\alpha\theta\big[\frac{\sqrt{2\pi}}{\omega}\hat{g}_0(((j-\tfrac{1}{2})\alpha)i)\\&\quad+(-1)^{j+1}\frac{2}{2j-1}\sqrt{\frac{2}{\pi}}\hat{h}_1(((j-\tfrac{1}{2})\alpha-1)i)\big]\\=&\Lambda_j\{f,g_0,h_1\}r^{(j-\frac{1}{2})\alpha}\sin(j-\tfrac{1}{2})\alpha\theta,\end{aligned}\tag{5}$$

where

$$\Lambda_j\{f,g_0,h_1\}=\Lambda'_{j,0}(g_0)+(-1)^{j+1}\Lambda'_{j,1}(h_1)+\Lambda''_j(f),\tag{6a}$$

and where the linear functionals $\Lambda'_{j,0}$, $\Lambda'_{j,1}$, and Λ''_j are given by

$$\begin{aligned}\Lambda'_{j,0}(g)&=\frac{\sqrt{2\pi}}{\omega}\hat{g}((j-\tfrac{1}{2})\alpha i),\\\Lambda'_{j,1}(h)&=\frac{2}{2j-1}\sqrt{\frac{2}{\pi}}\hat{h}(((j-\tfrac{1}{2})\alpha-1)i),\\\Lambda''_j(f)&=\frac{2}{2j-1}\sqrt{\frac{2}{\pi}}\int_0^\omega\hat{f}(((j-\tfrac{1}{2})\alpha-2)i,\varphi)\sin(j-\tfrac{1}{2})\alpha\varphi d\varphi.\end{aligned}\tag{6b}$$

Calculation of $v_{0,k}$ in the case of distinct poles

To calculate the residues at the points $\zeta_{0,k}$ we proceed as in the cases of Dirichlet and Neuman boundary conditions and obtain

$$\begin{aligned}&\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)\frac{\cosh\zeta(\omega-\theta)\sinh\zeta\varphi}{\zeta\cosh\zeta\omega}e^{i\zeta\tau}\\&\qquad=\frac{i}{2\pi}\frac{\cos k(\omega-\theta)\sin k\varphi}{k\sin k\omega}e^{-k\tau}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-l-2}\varphi}{l!(k-l-2)!}[D^l_{x_1}D^{k-l-2}_{x_2}f(0,0)],\;\;k=2,\cdots,n-1,\end{aligned}$$

$$\begin{aligned}&\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)\frac{\sinh\zeta\theta\cosh\zeta(\omega-\varphi)}{\zeta\cosh\zeta\omega}e^{i\zeta\tau}\\&\qquad=\frac{i}{2\pi}\frac{\sin k\theta\cos k(\omega-\varphi)}{k\sin k\omega}e^{-k\tau}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-l-2}\varphi}{l!(k-l-2)!}[D^l_{x_1}D^{k-l-2}_{x_2}f(0,0)],\;\;k=2,\cdots,n-1,\end{aligned}$$

$$\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{g}_0(\zeta)\frac{\cosh\zeta(\omega-\theta)}{\zeta\cosh\zeta\omega}e^{i\zeta\tau}=\frac{-i}{\sqrt{2\pi}}\frac{1}{k!}\frac{\cos k(\omega-\theta)}{k\cos k\omega}e^{-k\tau}g_0^{(k)}(0),\;\;k=0,\cdots,n-1,$$

$$\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{h}_1(\zeta-i)\frac{\sinh\zeta\theta}{\zeta\cosh\zeta\omega}e^{i\zeta\tau}=\frac{i}{\sqrt{2\pi}}\frac{1}{(k-1)!}\frac{\sin k\theta}{k\cos k\omega}e^{-k\tau}h_1^{(k-1)}(0),\;\;k=1,\cdots,n-1.$$

If $k = 0$ or 1, the right side of the formulas involving $\hat{f}$ must be replaced by 0, and if $k = 0$, the right side of the formulas involving $\hat{h}_1$ must be replaced by 0.

The function $v^*_{0,k}$ is calculated from the formula

$$v^*_{0,k} = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \cdot \lim_{\zeta \to \zeta_{0,k}} \quad = \sqrt{2\pi} i \cdot \lim_{\zeta \to \zeta_{0,k}} \quad .$$

Since $e^{-k\tau} = r^k$, we therefore obtain

$$\begin{aligned} v_{0,1}(\tau, \theta) =& \frac{r \cos(\omega - \theta)}{\cos \omega} g_0'(0) + \frac{r \sin \theta}{\cos \omega} h_1(0), \\ v_{0,k}(\tau, \theta) =& - \frac{r^k \cos k(\omega - \theta)}{k \cos k\omega} \sum_{l=0}^{k-2} \frac{D^l_{x_1} D^{k-l-2}_{x_2} f(0,0)}{l!(k-l-2)!} \int_0^\theta \sin k\varphi \cos^l \varphi \sin^{k-l-2} \varphi d\varphi \\ & - \frac{r^k \sin k\theta}{k \cos k\omega} \sum_{l=0}^{k-2} \frac{D^l_{x_1} D^{k-l-2}_{x_2} f(0,0)}{l!(k-l-2)!} \int_\theta^\omega \cos k(\omega - \varphi) \cos^l \varphi \sin^{k-l-2} \varphi d\varphi \\ & + \frac{r^k \cos k(\omega - \theta)}{k! \cos k\omega} g_0^{(k)}(0) + \frac{r^k \sin k\theta}{k! \cos k\omega} h_1^{(k-1)}(0), \;\; k = 2, \cdots, n-1. \end{aligned}$$

Evidently the coefficients of $g_0^{(k)}(0)$ and $h_1^{(k-1)}(0)$ are homogeneous polynomials of degree k in x. As in §II.2.2, we may show that

$$\begin{aligned} P(x) =& r^k \cos k(\omega - \theta) \int_0^\theta \sin k\varphi \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \\ & + r^k \sin k\theta \int_\theta^\omega \cos k(\omega - \varphi) \cos^l \varphi \sin^{k-2-l} \varphi d\varphi \end{aligned}$$

is a homogeneous polynomial of degree k in x. We conclude that

$$v_{0,k}(x) = \sum_{l=0}^{k-2} [D^l_{x_1} D^{k-2-l}_{x_2} f(0,0)] P_{k,l}(x) + g_0^{(k)}(0) P_k^{(0)}(x) + h_1^{(k-1)}(0) P_k^{(1)}(x), \tag{7}$$

where $P_{k,l}$, $P_k^{(0)}$, and $P_k^{(1)}$ are homogeneous polynomials in x of degree k.

Calculation of $v_{j,0}$ in the confluent case $\zeta_{j,0} = \zeta_{0,k}$.

We suppose that $(j - \frac{1}{2})\alpha = k$ for some integers j and k. In this case, the integrand has a double pole at $\zeta_{j,0}$. We employ a calculation similar to that in §II.2.2 to obtain

$$v_{j,0}(x) = \Lambda_j\{f, g_0, h_1\} [\theta \cos k\theta + (\ln r) \sin k\theta] r^k, \;\; (j - \tfrac{1}{2})\alpha = k = \text{integer}. \tag{8}$$

where $\Lambda_j\{f, g_0, h_1\}$ is defined by (6a) with $\Lambda'_{j,0}$, $\Lambda'_{j,1}$, and Λ''_j given by

$$\Lambda'_{j,0}(g) = C_{j,0} g^{(k)}(0), \;\; k = (j - \tfrac{1}{2})\alpha = \text{integer}, \tag{9a}$$

$$\Lambda'_{j,1}(h) = C_{j,1} h^{(k-1)}(0), \;\; k = (j - \tfrac{1}{2})\alpha = \text{integer}, \tag{9b}$$

$$\Lambda''_j(f) = \sum_{l=0}^{k-2} C_{k,l} [D^l_{x_1} D^{k-2-l}_{x_2} f(0,0)], \;\; k = (j - \tfrac{1}{2})\alpha = \text{integer}, \tag{10}$$

and where $v_{0,k}(x)$ is a homogeneous polynomial of degree k.

The following theorem is an immediate consequence of Lemma 1 and the above formulas.

Theorem 1. *Suppose* $\{f, g_0, h_1\} \in \mathcal{Y}_M^s(S)$ *with* $s > 2$, $s \neq$ *integer,* $2(s-1)/\alpha \neq$ *odd integer. Let* u *be the solution of (1), and suppose* $u \equiv 0$ *for* $r > 1$*. Then* $u(x) = V(x, s-1) + w(x, s-1)$ *where* w *is given by (4), where*

$$V(x, s-1) = \sum_{j=1}^{J(s)} v_{j,0}(x) + \sum_{\substack{k=1 \\ 2k/\alpha \neq \mathrm{odd integer}}}^{[s]-1} v_{0,k}(x),$$

and where the functions $v_{j,0}$ *and* $v_{0,k}$ *are given by (5), (7), and (8). The functions* $V(x, s-1)$ *and* $w(x, s-1)$*, regarded as functions of* (x, ω) *with* $x \in S(\omega)$*, are jointly continuous in these variables. The function* w *satisfies the conditions given in Lemma 1. In particular, for each* $a > 0$, $w \in H^s(S_a)$ *and*

$$||w(\cdot, s-1)||_{H^s(S_a)} \leq C(a)||\{f, g_0, h_1\}||_{\mathcal{Y}_M^s(S)}, \tag{11}$$

where S_a *is the truncated sector of radius* a*. Finally,* $u \in H^s(S)$ *if and only if either* $J(s) = 0$ *or* $\Lambda_j\{f, g_0, h_1\} = 0$ *for* $j = 1, \cdots, J(s)$ *and, if* $j\alpha = k =$ *integer,* $D^{k-2}f(0,0) = 0$, $g_0^{(k)}(0) = g_1^{(k)}(0) = 0$.

In Theorem 1, the hypothesis $s \neq$ integer is needed because $\hat{f}(\zeta - 2i, \varphi)$ $\hat{g}_l(\zeta)$ and $\hat{h}_l(\zeta - i)$may have poles when $\Im\zeta =$ integer, so $w^*(\tau, \theta, s-1)$ may not be defined when $s =$ integer. On the other hand, the singularities in the solution u do not arise from the poles $\zeta_{0,k}$, unless they are confluent poles. It is natural to seek a decomposition of the solution in the case $\{f, g_0, h_1\} \in \mathcal{Y}_M^s(S)$ for $s =$ integer. This is done in the next theorem.

Let

$$v_j(x) = \begin{cases} r^{(j-\frac{1}{2})\alpha} \sin(j - \frac{1}{2})\alpha\theta, & (j - \frac{1}{2})\alpha \neq \text{integer}, \\ [\theta \cos k\theta + (\ln r) \sin k\theta] r^k, & (j - \frac{1}{2})\alpha = k = \text{integer}. \end{cases} \tag{12}$$

$$v(x, s-1) = \sum_{1}^{J(s)} \Lambda_j\{f, g_0, h_1\} v_{j,0}(x), \tag{13}$$

$$P(x, s-1) = \sum_{k=1}^{[s]-1} v_{0,k}(x). \tag{14}$$

Thus, we have

$$V(x, s-1) = v(x, s-1) + P(x, s-1),$$

and we define

$$W(x, s-1) = w(x, s-1) + P(x, s-1).$$

Using these functions, there is obtained two representations of u:

$$\begin{aligned} u(x) &= v(x, s-1) + P(x, s-1) + w(x, s-1) \\ &= v(x, s-1) + W(x, s-1) && (15a) \\ &= V(x, s-1) + w(x, s-1). && (15b) \end{aligned}$$

The properties of the representation (15b) are established in Theorem 1, and the properties of (15a) are established in Theorem 2 below. Each of the representations (15a) and (15b) have advantages and disadvantages. The principal advantage of the representation (15a) lies in the fact, established in Theorem 2, that this representation is also valid when $s =$ integer. Also, (15a) clearly displays the fact that $u \in H^s(S)$ if and only if all the linear functionals $\Lambda_j\{f, g_0, h_1\} = 0$. On the other hand, the linear functionals Λ_j become infinite when ω is varied so that $j\alpha \to$ integer, and the function $v(x, s-1)$ is not continuous as a function of ω. The representation (15b) remedies this defect. The function $V = v + P$ is continuous in ω, the singularity in v being balanced by a singularity in P. This advantage comes at the cost of a greater complexity in the singular expansion V.

In the next theorem, the case $s =$ integer is allowed.

Theorem 2. *Suppose* $\{f, g_0, h_1\} \in \mathcal{Y}^s_M(S)$ *with* $s \geq 2$, $2(s-1)/\alpha \neq$ *odd integer. Let* u *be the solution of (1), and suppose* $u \equiv 0$ *for* $r > 1$. *Then*

$$u(x) = \sum_1^{J(s)} \Lambda_j\{f, g_0, h_1\} v_j(x) + W(x, s-1),$$

where $\Lambda_j\{f, g_0, h_1\}$, $j = 1, \cdots, J(s)$ *are bounded linear functionals on* $\mathcal{Y}^s_M(S)$, v_j *is given by (5), and, for each* $a > 0$, $W \in H^s(S_a)$. *There is a constant* $C(a) > 0$ *such that*

$$||W(\cdot, s-1)||_{H^s(S_a)} \leq C(a)||\{f, g_0, h_1\}||_{\mathcal{Y}^s_M(S)}. \tag{16}$$

Proof. If $s > 2$ and $s \neq$ integer, the result follows from Theorem 1. Suppose $s = k \geq 2$ is an integer, and $2(s-1)/\alpha \neq$ odd integer. Let $\varepsilon > 0$ be such that $j\alpha \neq (k-\varepsilon, k+\varepsilon)$ for $j = 1, \cdots$. Thus, $J(s-\varepsilon) = J(s+\varepsilon)$, so $v(x, k-\varepsilon-1) = v(k+\varepsilon-1)$, and therefore, $W(x, k-\varepsilon-1) = W(x, k+\varepsilon-1)$. We define $W(x, k-1) = W(x, k-\varepsilon-1)$. From Theorem 1, for each $a > 0$,

$$||W(\cdot, k-\varepsilon-1)||_{H^{k-\varepsilon}(S_a)} \leq C(a)||\{f, g_0, h_1\}||_{\mathcal{Y}^{k-\varepsilon}_M(S)},$$

$$||W(\cdot, k+\varepsilon-1)||_{H^{k+\varepsilon}(S_a)} \leq C(a)||\{f, g_0, h_1\}||_{\mathcal{Y}^{k+\varepsilon}_M(S)}.$$

Hence, by interpolation,

$$||W(\cdot, k-1)||_{H^k(S_a)} \leq C(a)||\{f, g_0, h_1\}||_{\mathcal{Y}^k_M(S)}.$$

If $2 < s \leq 1+\alpha$, $J(s) = 0$, $u = W(\cdot, s-1)$, and the bound for u follows from (11). In the case $s = 2$, the proof is a little different. Let $\{f, g_0, h_1\} \in \mathcal{Y}^2_M(S)$, let $\varepsilon > 0$, and let $\{f^\mu, g^\mu_0, h^\mu_1\} \in \mathcal{Y}^{2+\varepsilon}_M(S)$ be a sequence with $||\{f^\mu - f, g^\mu_0 - g_0, g^\mu_1 - g_1\}||_{\mathcal{Y}^2_M(S)} \to 0$ as $\mu \to \infty$. Let u^μ be the solution of (1) with data $\{f^\mu, g^\mu_0, h^\mu_1\}$. Suppose, without loss of generality, that $u^\mu(x) = 0$ for $r > 1$. Suppose ε is so small that $2+\varepsilon < (J(2)+1)\alpha$, so $J(2+\varepsilon) = J(2)$. Applying Theorem 2 with $s = 2+\varepsilon$, we have the decomposition

$$u^\mu(x) = v^\mu(x, s+\varepsilon-1) + W^\mu(x, s+\varepsilon-1).$$

The singular expansion v^μ is given by

$$v^\mu(x, s+\varepsilon-1) = \sum_1^{J(2)} \Lambda_j\{f^\mu, g^\mu_0, h^\mu_1\} v_j(x).$$

Since each Λ_j is a bounded linear functional on $\mathcal{Y}^2_M(S)$, $v^\mu \to v$ as $\mu \to \infty$ where

$$v(x, 1) = \sum_1^{J(2)} \Lambda_j\{f, g_0, h_1\} r^{j\alpha} \sin j\alpha\theta.$$

Since $\Delta v^\mu = 0$ and $v^\mu = 0$ on Γ_0, $v^\mu_n = 0$ on Γ_1, $\Delta W^\mu = f^\mu$ and $W^\mu = g^\mu_0$ on Γ_0, $W^\mu_n = h_1$ on Γ_1. From Theorem 2 in the case $s > 2$, $W^\mu \in H^{2+\varepsilon}(S_a)$. Hence we may apply (2.1;27) to $W^\mu - W^\nu$ to obtain $||W^\mu - W^\nu||_{H^2(S)} \to 0$ as $\mu, \nu \to \infty$. Hence W^μ converges in $H^2(S)$ to a function $W \in H^2(S)$. We therefore obtain the desired decomposition of u in the case $s = 2$. ■

Calculation of Λ_j *in the case of distinct poles*

In the confluent case, when $(j - \frac{1}{2})\alpha = k$ for some integers j and k, the linear functionals $\Lambda'_{j,0}(g)$, $\Lambda'_{j,1}(h)$, and $\Lambda''_j(f)$ are given by the formulas (9a,b) and (10). It is of interest to obtain formulas for these linear functionals when $(j-\frac{1}{2})\alpha \neq$ integer. For this, we must evaluate $\hat{f}(\eta i,\theta)$ for $\eta > -1$ $\hat{g}_0(\eta i)$, and $\hat{h}_1(\eta i)$. We have

$$
\begin{aligned}
\Lambda'_{j,0}(g) = &-\frac{1}{\omega}\sum_{0}^{n-1}\frac{g_0^{(k)}(0)}{((j-\frac{1}{2})\alpha-k)k!}\\
&+\frac{1}{\omega}\int_0^1 [g_0(x)-T_{0,n-1}(x)]x^{-(j-\frac{1}{2})\alpha-1}dx,\ (j-\tfrac{1}{2})\alpha\neq \text{integer},
\end{aligned}\tag{17a}
$$

$$
\begin{aligned}
\Lambda'_{j,1}(h) = &-\frac{1}{(2j-1)\pi}\sum_{0}^{n-2}\frac{h_1^{(k)}(0)}{((j-\frac{1}{2})\alpha-k-1)k!}\\
&+\frac{1}{(2j-1)\pi}\int_0^1 [h_1(x)-T_{1,n-2}(x)]x^{-(j-\frac{1}{2})\alpha-1}dx,\ (j-\tfrac{1}{2})\alpha\neq \text{integer},
\end{aligned}\tag{17b}
$$

$$
\begin{aligned}
\Lambda''_{j}(f) = &-\frac{1}{(2j-1)\pi}\sum_{0\le k+m\le n-3}\frac{A_{k,m}(\omega)}{k!m!((j-\frac{1}{2})\alpha-(k+m+2))}[D^k_{x_1}D^m_{x_2}f(0,0)]\\
&+\frac{1}{(2j-1)\pi}\int\int_{S_1} [f(x)-T_{n-3}(x)]r^{-(j-\frac{1}{2})\alpha}\cos(j-\tfrac{1}{2})\alpha\theta dx,\ (j-\tfrac{1}{2})\alpha\neq \text{integer},
\end{aligned}\tag{17c}
$$

where

$$A_{k,m}(\omega)=\int_0^\omega \cos^k\varphi\sin^m\varphi\sin(j-\tfrac{1}{2})\alpha\varphi d\varphi.$$

From the construction, if $(j-\frac{1}{2})\alpha \neq$ integer, $\{f,g_0,h_1\}\to\Lambda'_{j,0}(g_0)$, $\{f,g_0,h_1\}\to\Lambda'_{j,1}(h_1)$, and $\{f,g_0,h_1\}\to\Lambda''_j(f)$ are bounded linear functionals on X^s. We verify this directly. Since $g_0\in H^{s-\frac{1}{2}}(R_+)$ and $s=n+\sigma$ with $0<\sigma<1$, g_0 is $(n-1)$-times continuously differentiable on R_+, so $g_0\to g_0^{(k)}(0)$ is a bounded linear functional on $H^{s-\frac{1}{2}}(R_+)$ if $k\neq n-1$. Also, from Theorem 1.1;6,

$$\int_0^1 x^{-2(s-1/2)}|g_0-T_{0,n-1}|^2dx\le C||g_0||^2_{s-1/2}.$$

Since $(j-\frac{1}{2})\alpha<s-1$,

$$
\begin{aligned}
|\int_0^1[g_0(x)-T_{0,n-2}(x)]x^{-(j-\frac{1}{2})\alpha}dx| \le &\int_0^1 x^{-(s-1/2)}|g_0(x)-T_{0,n-2}(x)|\cdot x^{s-3/2-(j-\frac{1}{2})\alpha}dx\\
\le&\{\int_0^1 x^{2(s-(j-\frac{1}{2})\alpha-3/2)}dx\}^{1/2}||g_0||_{s-1/2}\\
\le&C||g_0||_{s-1/2}.
\end{aligned}
$$

Hence $\Lambda'_{M,0,j}(g_0)\le C||g_0||_{s-1/2}\le||\{f,g_0,h_1\}||_{X^s}$. The other two linear functionals are bounded in a similar manner.

Another mixed problem on S is obtained by putting the Dirichlet boundary condition on Γ_1 and putting the Neumann boundary condition on Γ_0. Thus, we may consider (1a) with the boundary conditions

$$-u_y(x,0)=h_0(x),\ x>0,\tag{1b'}$$

$$u(r\cos\omega,r\sin\omega)=g_1(r),\ r>0.\tag{1c'}$$

The two problems are related by a change of independent variable, $\theta'=\omega-\theta$. Upon making this change of variable, one obtains singular functions v_j for the problem (1a,b',c') given by the formula

$$
v_j(x)=\begin{cases} r^{(j-\frac{1}{2})\alpha}\cos(j-\frac{1}{2})\alpha\theta, & (j-\frac{1}{2})\alpha\neq\text{integer},\\ [\theta\sin(j-\frac{1}{2})\alpha\theta+(\ln r)\cos(j-\frac{1}{2})\alpha\theta]r^{(j-\frac{1}{2})\alpha}, & (j-\frac{1}{2})\alpha=\text{integer}.\end{cases}\tag{8$'$}
$$

With these functions, Theorem 1 and Theorem 2 are valid for the problem (1a,b',c').