## 2.3 The equation $-\Delta u = f$ in a sector - Neumann conditions

This section treats the Neumann problem

$$
\begin{aligned}
&(1a) && -\Delta u = f \text{ in } S,\\
&(1b) && -u_y(x,0) = h_0(x),\ x>0\\
& && D_\nu u(r\cos\omega, r\sin\omega) := -(\sin\omega)u_x(r\cos\omega, r\sin\omega)\\
&(1c) && \qquad + (\cos\omega)u_y(r\cos\omega, r\sin\omega) = h_1(r),\ r>0.
\end{aligned}
$$

in a sector $S$ with angle $\omega$. The data $\{f, h_0, h_1\}$ is assumed to have a certain amount of regularity, and is assumed to satisfy the compatibility condition

$$
(2) \qquad \int\int_S f dx + \int_0^\infty [h_0(x) + h_1(x)]dx = 0.
$$

There is obtained an expansion of a weak solution of (1) into a set of singular functions plus a smooth remainder.

We shall use the data space $\mathcal{Y}_N^s(S)$ defined in §II.1. That is, for $s \geq 2$, $\mathcal{Y}_N^s(S)$ is the subset of triples $\{f, h_0, h_1\} \in H^{s-2}(S) \times H^{s-3/2}(\Gamma) \times H^{s-3/2}(\Gamma)$ which satisfy (2). A function $u \in H^1(S)$ is defined to be a weak solution of (1) if $u$ satisfies

$$
\int\int_S \nabla u \cdot \nabla\varphi dx = \int_0^\infty [h_0(r)\varphi(r,0) + h_1(r)\varphi(r\cos\omega, r\sin\omega)]dr + \int\int_S f\varphi dx \text{ for } \varphi \in H^1(S).
$$

We shall consider weak solutions which also satisfy

$$
u(x) \equiv 0 \text{ for } r = \sqrt{x_1^2 + x_2^2} > 1.
$$

If $s \geq 2$ and if $\omega \neq \pi$ or $\omega \neq 2\pi$, for any pair $\{h_0, h_1\} \in H^{s-3/2}(R_+) \times H^{s-3/2}(R_+)$, there is a function $H \in H^s(S)$ such that $-H_y = h_0$ on $\Gamma_0$ and $D_\nu H = h_1$ on $\Gamma_1$. Thus, there is no issue of incompatibility of the boundary data at 0 in these cases. If $\omega = \pi$, the sector $S$ is the half plane $R_+^2$. The boundary conditions (1b) and (1c) may be interpreted as Neumann boundary conditions on $y = 0$ with a possible jump in the boundary data at the origin. In this case, in order for there to be a function $H \in H^s(S)$ such that $-H_y(x,0) = h_0(x)$ for $x > 0$ and $-H_y(x,0) = h_1(x)$ for $x < 0$, the Neumann data $\{h_0, h_1\} \in H^{s-3/2}(R_+) \times H^{s-3/2}(R_+)$ must satisfy the compatibility condition

$$
\int_0^\infty x^{-1}|h_0(x) - h_1(-x)|^2 dx < \infty \text{ if } s = 2,
$$

$$
h_0(0) = h_1(0) \text{ if } s > 2.
$$

Similar conditions hold in the case $\omega = 2\pi$. Here, $S$ is the slit plane $s = R^2 \setminus \{(x,0) : x > 0\}$. In this case, in order for there to be a function $H \in H^s(S)$ such that $-H_y(x,+0) = h_0(x)$ for $x > 0$ and $H_y(x,-0) = h_1(x)$ for $x > 0$, the Neumann data $\{h_0, h_1\} \in H^{s-3/2}(R_+) \times H^{s-3/2}(R_+)$ must satisfy the compatibility condition

$$
\int_0^\infty x^{-1}|h_0(x) - h_1(x)|^2 dx < \infty \text{ if } s = 2,
$$

$$
h_0(0) = h_1(0) \text{ if } s > 2.
$$

As will be seen, the singular expansions will introduce singular functions to remove these incompatibilities in the Neumann boundary data at the origin in the special cases $\omega = \pi$ and $\omega = 2\pi$.

Since $u \in H^s(S_1)$ for $0 < s < 1$, $\hat{u}(\zeta, \theta)$ is analytic in the half plane $\eta < 0$. From Lemma II.1;3, $\hat{u} = \mathcal{K}u$ satisfies the two point boundary value problem

$$
\begin{aligned}
&(3a) && -\hat{u}_{\theta\theta}(\zeta,\theta) + \zeta^2\hat{u}(\zeta,\theta) = \hat{f}(\zeta - 2i, \theta),\ \eta < 0,\\
&(3b) && \hat{u}_\theta(\zeta, 0) = -\hat{h}_0(\zeta - i),\ \hat{u}_\theta(\zeta,\omega) = \hat{h}_1(\zeta - i),\quad \eta < 0.
\end{aligned}
$$

The compatibility condition (2) transforms to

$$\int_0^\omega \hat{f}(-2i,\theta)\theta + \hat{h}_0(-i) + \hat{h}_1(-i) = 0. \tag{4}$$

The equation (4) is a consequence of (3a,b) at $\zeta = 0$.

The problem (3a,b) can be solved exactly. The solution is given by the formula

$$\begin{aligned}\hat{u}(\zeta,\theta) = &\int_0^\theta \hat{f}(\zeta-2i,\varphi)\frac{\cosh\zeta(\omega-\theta)\cosh\zeta\varphi}{\zeta\sinh\zeta\omega}d\varphi\\ &+\int_\theta^\omega \hat{f}(\zeta-2i,\varphi)\frac{\cosh\zeta\theta\cosh\zeta(\omega-\varphi)}{\zeta\sinh\zeta\omega}d\varphi\\ &+\hat{h}_0(\zeta-i)\frac{\cosh\zeta(\omega-\theta)}{\zeta\sinh\zeta\omega}+\hat{h}_1(\zeta-i)\frac{\cosh\zeta\theta}{\zeta\sinh\zeta\omega}.\end{aligned} \tag{5}$$

Note that each term in (5) becomes singular as $\zeta \to 0$. Because of the compatibility condition (4), the singularities in the various terms cancel, and a finite limit is obtained for $\hat{u}(0,\theta)$.

Now suppose $\{f, h_0, h_1\} \in \mathcal{Y}_N^s(S)$ with $s > 2$. Suppose $s \neq$ integer, and write $s = n + \sigma$ with $0 < \sigma < 1$. Note that $s - \frac{3}{2} = n - 1 + \sigma - \frac{1}{2}$, and $-\frac{1}{2} < \sigma - \frac{1}{2} < \frac{1}{2}$. Hence $n-1$ is the closest integer to $s - \frac{3}{2}$. Since $h_l \in H^{s-3/2}(R_+)$, from Theorem I.2;6, $\hat{h}_l(\zeta)$ has a meromorphic extension to the half plane $\eta < s-2$ given by

$$\hat{h}_l(\zeta) = \hat{T}_{l,n-2}(\zeta) + \hat{R}_{l,n-2}(\zeta),$$

where

$$\hat{T}_{l,n-2}(\zeta) = \frac{-i}{\sqrt{2\pi}}\sum_{k=0}^{n-2}\frac{h_l^{(k)}(0)}{k!}\frac{1}{\zeta-ki},$$

and where $\hat{R}_{l,n-2}(\zeta)$ is analytic in $\eta < s-2$ and is given by

$$\hat{R}_{l,n-2}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty [h_l^*(\tau) - T_{l,n-2}^*(\tau)]e^{-i\zeta\tau}d\tau.$$

Similarly, since $f \in H^{s-2}(S)$, Theorem I.4;6 implies that $\hat{f}(\zeta,\theta)$ has a meromorphic extension to the half plane $\eta < s-3$ given by

$$\hat{f}(\zeta,\theta) = \hat{T}_{n-3}(\zeta,\theta) + \hat{R}_{n-3}(\zeta,\theta).$$

where

$$\hat{T}_{n-3}(\zeta,\theta) = \frac{-i}{\sqrt{2\pi}}\sum_{0\le k+m\le n-3}\frac{\cos^k\theta\sin^m\theta}{k!m!}[D_{x_1}^k D_{x_2}^m f(0,0)]\frac{1}{\zeta-(k+m)i},$$

and where $\hat{R}_{n-3}(\zeta,\theta)$ is analytic in $\zeta < s-3$ and is given by

$$\hat{R}_{n-3}(\zeta,\theta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty [f^*(\tau,\theta) - T_{l,n-2}^*(\tau,\theta)]e^{-i\zeta\tau}d\tau.$$

Suppose $s$ is chosen so that $s \neq$ integer and $(s-1)/\alpha \neq$ integer. Then the integrands in (3) are finite on $\eta = s-1$, and are meromorphic in $\eta < s-1$ with poles at the points $\zeta_{j,0} = j\alpha i$, $j = 1, 2, \cdots, J(s)$, and at the points $\zeta_{0,k} = ki$, $k = 0, 1, \cdots, n-1$. Here we define

$$J(s) = \max\{j : j\alpha < s-1\}.$$

For $\eta \le s-1$, $\eta \neq$ integer, $\eta/\alpha \neq$ integer, define a function $w^*(\tau,\theta,\eta)$ by

$$\begin{aligned}w^*(\tau,\theta,\eta) = &\frac{1}{\sqrt{2\pi}}\int_0^\theta\int_{\xi=-\infty}^\infty \hat{f}(\zeta-2i,\varphi)\frac{\cosh\zeta(\omega-\theta)\cosh\zeta\varphi}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}d\xi d\varphi\\ &+\frac{1}{\sqrt{2\pi}}\int_\theta^\omega\int_{\xi=-\infty}^\infty \hat{f}(\zeta-2i,\varphi)\frac{\cosh\zeta\theta\cosh\zeta(\omega-\varphi)}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}d\xi d\varphi\\ &+\frac{1}{\sqrt{2\pi}}\int_{\xi=-\infty}^\infty \hat{h}_0(\zeta-i)\frac{\cosh\zeta(\omega-\theta)}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}d\xi+\frac{1}{\sqrt{2\pi}}\int_{\xi=-\infty}^\infty \hat{h}_1(\zeta-i)\frac{\cosh\zeta\theta}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}d\xi,\end{aligned} \tag{6}$$

where the integral is taken on the line $\Im\zeta = \eta$. The integrals in (6) are well defined by the conditions on $s$. We define $V^*(\tau,\theta,\eta) = u^*(\tau,\theta) - w^*(\tau,\theta,\eta)$. As usual, we let $V(x,\eta)$ and $w(x,\eta)$ denote the corresponding functions in the $x$-variable. The following lemma gives some properties of the decomposition $u = V + w$.

**Lemma 1.** *Suppose $\{f, h_0, h_1\} \in \mathcal{Y}_N^s(S)$ with $s > 2$, $s \neq$ integer, $(s-1)/\alpha \neq$ integer. Let $u$ be the solution of (1), and suppose $u \equiv 0$ for $r > 1$. Then $u(x) = V(x, s-1) + w(x, s-1)$ where $w$ is given by (6), and where $V$ is the sum of the residues of the integrals in (6) arising from the poles of the integrands in the strip $0 < \Im\zeta < s-1$. The functions $V(x, s-1)$ and $w(x, s-1)$, regarded as functions of $(x,\omega)$ with $x \in S(\omega)$, are jointly continuous in these variables. For each $a > 0$ and $j = 0, \cdots, n$, $r^{j-s}D^j w \in H^0(S_a)$ and $r^{j-n}D^j w \in H^\sigma(S_a)$, where $S_a$ is the truncated sector of radius $a$. There is a constant $C(s,a) > 0$ such that*

$$||r^{j-s}D^j w(\cdot, s-1)||_{0,S_a} \leq C(a)||\{f, h_0, h_1\}||_{\mathcal{Y}_N^s(S)},$$

$$||r^{j-n}D^j w(\cdot, s-1)||_{\sigma,S_a} \leq C(a)||\{f, h_0, h_1\}||_{\mathcal{Y}_N^s(S)}.$$

*Also,*

$$D_x^m w(0, s-1) = 0 \text{ for } 0 \leq m \leq \lfloor s \rfloor - 2.$$

Proof. Since $\hat{u}$ solves (3),

$$u^*(\tau,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\zeta,\theta) e^{i\zeta\theta} d\xi = w^*(\tau\theta\eta) \text{ for } \eta < 0.$$

We move the line of integration in (6) from $\Im\zeta = \eta < 0$ to $\Im\zeta = s-1$. Using the Cauchy integral formula and an estimate for some integrals on the lines $\Re\zeta = \pm\Xi$, it is seen that $V^*(\tau,\theta,s-1)$ is the sum of residues of the four integrals in (6) at the poles. The continuity of $V$ and $w$ as a function of $\omega$ follows from the fact that the location of the poles and residues of the poles are continuous functions of $\omega$. The proof of the remaining properties of $w$ follows the corresponding portion of the proof of Theorem 2.2.2;1. ■

We now calculate the function $V(x, s-1)$. Write

$$V(x, s-1) = \sum_1^{J(s)} v_{j,0}(x) + \sum_0^{[s]-1} v_{0,k}(x),$$

where $v_{j,0}(x)$ is the contribution to $V$ arising from the pole at $\zeta_{j,0}$, and $v_{0,k}(x)$ is the contribution to $V$ arising from the pole at $\zeta_{0,k}$. The calculation is divided into the cases when the poles $\zeta_{j,0}$ and $\zeta_{0,k}$ are distinct, and the confluent case when $\zeta_{j,0} = \zeta_{0,k}$ for some $j$ and $k$,

*Calculation of $v_{j,0}$ in the case of distinct poles*

To calculate the residues at the points $\zeta_{j,0}$ we use the formula $\sinh\zeta\omega \approx (-1)^j\omega(\zeta - j\alpha i)$ near $\zeta_{j,0}$. We obtain

$$\lim_{\zeta\to\zeta_{j,0}} (\zeta - \zeta_{j,0})\hat{f}(\zeta - 2i, \varphi) \frac{\cosh\zeta(\omega-\theta)\cosh\zeta\varphi}{\zeta\sinh\zeta\omega} e^{i\zeta\tau} = \frac{(-1)^{j+1}i}{j\pi} \hat{f}((j\alpha - 2)i, \varphi) \cos j\alpha(\omega-\theta)\cos j\alpha\varphi e^{-j\alpha\tau},$$

$$\lim_{\zeta\to\zeta_{j,0}} (\zeta - \zeta_{j,0})\hat{f}(\zeta - 2i, \varphi) \frac{\cosh\zeta\theta\cosh\zeta(\omega-\varphi)}{\zeta\sinh\zeta\omega} e^{i\zeta\tau} = \frac{(-1)^{j+1}i}{j\pi} \hat{f}((j\alpha - 2)i, \varphi) \cos j\alpha\theta\cos j\alpha(\omega-\varphi) e^{-j\alpha\tau},$$

$$\lim_{\zeta\to\zeta_{j,0}} (\zeta - \zeta_{j,0})\hat{h}_0(\zeta - i) \frac{\cosh\zeta(\omega-\theta)}{\zeta\sinh\zeta\omega} e^{i\zeta\tau} = (-1)^{j+1}\frac{i}{j\pi}\hat{h}_0((j\alpha-1)i)\cos j\alpha(\omega-\theta)e^{-j\alpha\tau},$$

$$\lim_{\zeta\to\zeta_{j,0}} (\zeta - \zeta_{j,0})\hat{h}_1(\zeta - i) \frac{\cosh\zeta\theta}{\zeta\sinh\zeta\omega} e^{i\zeta\tau} = (-1)^{j+1}\frac{i}{j\pi}\hat{h}_1((j\alpha-1)i)\cos j\alpha\theta e^{-j\alpha\tau},$$

The function $v_{j,0}^*$ is calculated from the formula

$$v_{j,0}^* = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \cdot \lim_{\zeta\to\zeta_{j,0}} = \sqrt{2\pi} i \cdot \lim_{\zeta\to\zeta_{j,0}}.$$

Using this and the fact that $\cos j\alpha(\omega-\theta)=(-1)^j\cos j\alpha\theta$, we obtain

$$
\begin{aligned}
v_{j,0}(x) =&\frac{1}{j}\sqrt{\frac{2}{\pi}}r^{j\alpha}\cos j\alpha\theta\int_0^{\omega}\hat{f}((j\alpha-2)i,\varphi)\cos j\alpha\varphi d\varphi \\
&+\frac{1}{j}\sqrt{\frac{2}{\pi}}r^{j\alpha}\cos j\alpha\theta[\hat{h}_0((j\alpha-1)i)+(-1)^j\hat{h}_1((j\alpha-1)i)] \\
=&\Lambda_j\{f,h_0,h_1\}r^{j\alpha}\cos j\alpha\theta,
\end{aligned}
\tag{7}
$$

where

$$
\Lambda_j\{f,h_0,h_1\}=\Lambda_j'(h_0)+(-1)^j\Lambda_j'(h_1)+\Lambda_j''(f),
\tag{8a}
$$

and where the linear functionals $\Lambda_j'$ and $\Lambda_j''$ are given by

$$
\begin{aligned}
\Lambda_j'(h)&=\frac{1}{j}\sqrt{\frac{2}{\pi}}\hat{h}((j\alpha-1)i),\\
\Lambda_j''(f)&=\frac{1}{j}\sqrt{\frac{2}{\pi}}\int_0^{\omega}\hat{f}((j\alpha-2)i,\varphi)\cos j\alpha\varphi d\varphi.
\end{aligned}
\tag{8b}
$$

*Calculation of $v_{0,k}$ in the case of distinct poles*

To calculate the residues at the points $\zeta_{0,k}$, we use the formulas

$$
\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)=\frac{-i}{\sqrt{2\pi}}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-l-2}\varphi}{l!(k-l-2)!}[D_{x_1}^lD_{x_2}^{k-l-2}f(0,0)],\ k=2,\cdots,n-1,
$$

$$
\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{h}_l(\zeta-i)=\frac{-i}{\sqrt{2\pi}}\frac{1}{(k-1)!}h_l^{(k-1)}(0),\ k=1,\cdots,n-1.
$$

We therefore obtain

$$
\begin{aligned}
\lim_{\zeta\to\zeta_{0,k}}&(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)\frac{\cosh\zeta(\omega-\theta)\cosh\zeta\varphi}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}\\
&=\frac{i}{\sqrt{2\pi}}\frac{\cos k(\omega-\theta)\cos k\varphi}{k\sin k\omega}e^{-k\tau}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-l-2}\varphi}{l!(k-l-2)!}[D_{x_1}^lD_{x_2}^{k-l-2}f(0,0)],
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\zeta\to\zeta_{0,k}}&(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)\frac{\cosh\zeta\theta\cosh\zeta(\omega-\varphi)}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}\\
&=\frac{i}{\sqrt{2\pi}}\frac{\cos k\theta\cos k(\omega-\varphi)}{k\sin k\omega}e^{-k\tau}\sum_{l=0}^{k-2}\frac{\cos^l\varphi\sin^{k-l-2}\varphi}{l!(k-l-2)!}[D_{x_1}^lD_{x_2}^{k-l-2}f(0,0)],
\end{aligned}
$$

$$
\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{h}_0(\zeta-i)\frac{\cosh\zeta(\omega-\theta)}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}=\frac{i}{\sqrt{2\pi}}\frac{1}{(k-1)!}\frac{\cos k(\omega-\theta)}{k\sin k\omega}e^{-k\tau}h_0^{(k-1)}(0),
$$

$$
\lim_{\zeta\to\zeta_{0,k}}(\zeta-\zeta_{0,k})\hat{h}_1(\zeta-i)\frac{\cosh\zeta\theta}{\zeta\sinh\zeta\omega}e^{i\zeta\tau}=\frac{i}{\sqrt{2\pi}}\frac{1}{(k-1)!}\frac{\cos k\theta}{k\sin k\omega}e^{-k\tau}h_1^{(k-1)}(0).
$$

If $k=0$ or $1$, the first two formulas must be modified since $\hat{f}$ is analytic at $\zeta=0$ and $\zeta=-1$, so $\lim(\zeta-\zeta_{0,k})\hat{f}(\zeta-2i,\varphi)=0$ for $k=0,1$. Similarly, in the case $k=0$, the last two formulas must be modified, since $\hat{h}_l$ is analytic at $\zeta=0$, so $\lim(\zeta-\zeta_{0,k})\hat{h}_l(\zeta-i)=0$ for $k=0$.

The function $v^*_{0,k}$ is calculated from the formula

$$v^*_{0,k} = \frac{1}{\sqrt{2\pi}} \cdot 2\pi i \cdot \lim_{\zeta \to \zeta_{0,k}} = \sqrt{2\pi} i \cdot \lim_{\zeta \to \zeta_{0,k}} .$$

Since $e^{-k\tau} = r^k$ we obtain

$$\begin{aligned}
v_{0,1}(x) =& \frac{r\cos(\omega-\theta)}{\sin\omega} h_0(0) + \frac{r\cos\theta}{\sin\omega} h_1(0),\\
v_{0,k}(x) =& -\frac{r^k \cos k(\omega-\theta)}{k \sin k\omega} \sum_{l=0}^{k-2} \frac{D^l_{x_1} D^{k-l-2}_{x_2} f(0,0)}{l!(k-l-2)!} \int_0^\theta \cos k\varphi \cos^l \varphi \sin^{k-l-2}\varphi d\varphi\\
&- \frac{r^k \cos k\theta}{k \sin k\omega} \sum_{l=0}^{k-2} \frac{D^l_{x_1} D^{k-l-2}_{x_2} f(0,0)}{l!(k-l-2)!} \int_\theta^\omega \cos k(\omega-\varphi) \cos^l \varphi \sin^{k-l-2}\varphi d\varphi\\
&+ \frac{r^k \cos k(\omega-\theta)}{k! \sin k\omega} h_0^{(k-1)}(0) + \frac{r^k \cos k\theta}{k! \sin k\omega} h_1^{(k-1)}(0),\ \ k = 2, \cdots, n-1.
\end{aligned}$$

Evidently the coefficients of $h_0^{(k-1)}(0)$ and $h_1^{(k-1)}(0)$ are homogeneous polynomials of degree $k$ in $x$. As in §II.2.2, we may show that

$$\begin{aligned}
P(x) =& r^k \cos k(\omega-\theta) \int_0^\theta \cos k\varphi \cos^l \varphi \sin^{k-2-l}\varphi d\varphi\\
&+ r^k \cos k\theta \int_\theta^\omega \cos k(\omega-\varphi) \cos^l \varphi \sin^{k-2-l}\varphi d\varphi
\end{aligned}$$

is a homogeneous polynomial of degree $k$ in $x$. We conclude that

$$v_{0,k}(x) = \sum_{l=0}^{k-2} [D^l_{x_1} D^{k-2-l}_{x_2} f(0,0)] P_{k,l}(x) + h_0^{(k-1)}(0) P_k^{(0)}(x) + h_1^{(k-1)}(0) P_k^{(1)}(x), \tag{9}$$

where $P_{k,l}$, $P_k^{(0)}$, and $P_k^{(1)}$ are homogeneous polynomials in $x$ of degree $k$.

*Calculation of $v_{j,0}$ in the confluent case $\zeta_{j,0} = \zeta_{0,k}$.*

We suppose that $j\alpha = k$ for some integers $j$ and $k$. In this case, the integrand has a double pole at $\zeta_{j,0}$. We employ a calculation similar to that in §II.2.2 to obtain

$$v_{j,0}(x) = \Lambda_j\{f, h_0, h_1\}[-(\ln r) r^k \cos k\theta + \theta r^k \sin k\theta] + v_{0,k}(x),\ \ k = j\alpha = \text{integer}, \tag{10}$$

where $\Lambda_j\{f, h_0, h_1\}$ is defined by (8a) with $\Lambda'_j$ and $\Lambda''_j$ given by

$$\Lambda'_j(h) = C_j h^{(k-1)}(0),\ \ k = j\alpha = \text{integer}, \tag{11}$$

$$\Lambda''_j(f) = \sum_{l=0}^{k-2} C_{k,l} [D^l_{x_1} D^{k-2-l}_{x_2} f(0,0)],\ \ k = j\alpha = \text{integer}, \tag{12}$$

and where $v_{0,k}(x)$ is a homogeneous polynomial of degree $k$.

The following theorem is an immediate consequence of Lemma 1 and the above formulas.

**Theorem 1.** *Suppose* $\{f, h_0, h_1\} \in \mathcal{Y}_N^s(S)$ *with* $s > 2$, $s \neq$ *integer,* $(s-1)/\alpha \neq$ *integer. Let* $u$ *be the solution of (1), and suppose* $u \equiv 0$ *for* $r > 1$. *Then* $u(x) = V(x, s-1) + w(x, s-1)$ *where* $w$ *is given by (6), where*

$$V(x, s-1) = \sum_{j=1}^{J(s)} v_{j,0}(x) + \sum_{\substack{k=1 \\ k/\alpha \neq \text{integer}}}^{[s]-1} v_{0,k}(x),$$

*and where the functions* $v_{j,0}$ *and* $v_{0,k}$ *are given by (7), (9), and (10). The functions* $V(x, s-1)$ *and* $w(x, s-1)$, *regarded as functions of* $(x, \omega)$ *with* $x \in S(\omega)$, *are jointly continuous in these variables. The function* $w$ *satisfies the conditions given in Lemma 1. In particular, for each* $a > 0$, $w \in H^s(S_a)$ *and*

$$||w(\cdot, s-1)||_{H^s(S_a)} \leq C(a)||\{f, h_0, h_1\}||_{\mathcal{Y}_N^s(S)}, \tag{13}$$

*where* $S_a$ *is the truncated sector of radius* $a$. *Finally,* $u \in H^s(S)$ *if and only if either* $J(s) = 0$ *or* $\Lambda_j\{f, h_0, h_1\} = 0$ *for* $j = 1, \cdots, J(s)$ *and, if* $j\alpha = k =$ *integer,* $D^{k-2}f(0,0) = 0$, $h_0^{(k-1)}(0) = h_1^{(k-1)}(0) = 0$.

In Theorem 1, the hypothesis $s \neq$ integer is needed because $\hat{f}(\zeta - 2i, \varphi)$ and $\hat{h}_l(\zeta - i)$ may have poles when $\Im\zeta =$ integer, so $w^*(\tau, \theta, s-1)$ may not be defined when $s =$ integer. On the other hand, the singularities in the solution $u$ do not arise from the poles $\zeta_{0,k}$, unless they are confluent poles. It is natural to seek a decomposition of the solution in the case $\{f, h_0, h_1\} \in \mathcal{Y}_N^s(S)$ for $s =$ integer. This is done in the next theorem.

Let

$$v_j(x) = \begin{cases} r^{j\alpha} \cos j\alpha\theta, & j\alpha \neq \text{integer}, \\ -(\ln r) r^k \cos k\theta + \theta r^k \sin k\theta, & j\alpha = k = \text{integer}, \end{cases} \tag{14}$$

$$v(x, s-1) = \sum_{1}^{J(s)} \Lambda_j v_{j,0}(x), \tag{15}$$

$$P(x, s-1) = \sum_{k=1}^{[s]-1} v_{0,k}(x), \tag{16}$$

Thus, we have

$$V(x, s-1) = v(x, s-1) + P(x, s-1),$$

and we define

$$W(x, s-1) = w(x, s-1) + P(x, s-1).$$

Using these functions, there is obtained two representations of $u$:

$$\begin{aligned} u(x) &= v(x, s-1) + P(x, s-1) + w(x, s-1) \\ &= v(x, s-1) + W(x, s-1) && (17a) \\ &= V(x, s-1) + w(x, s-1). && (17b) \end{aligned}$$

The properties of the representation (17b) are established in Theorem 1, and the properties of (17a) are established in Theorem 2 below. Each of the representations (17a) and (17b) have advantages and disadvantages. The principal advantage of the representation (17a) lies in the fact, established in Theorem 2, that this representation is also valid when $s =$ integer. Also, (17a) clearly displays the fact that $u \in H^s(S)$ if and only if all the linear functionals $\Lambda_j\{f, h_0, h_1\} = 0$. On the other hand, the linear functionals $\Lambda_j$ become infinite when $\omega$ is varied so that $j\alpha \to$ integer, and the function $v(x, s-1)$ is not continuous as a function of $\omega$. The representation (17b) remedies this defect. The function $V = v + P$ is continuous in $\omega$, the singularity in $v$ being balanced by a singularity in $P$. This advantage comes at the cost of a greater complexity in the singular expansion $V$.

In the next theorem, the case $s =$ integer is allowed.

**Theorem 2.** *Suppose $\{f, h_0, h_1\} \in \mathcal{Y}_N^s(S)$ with $s \geq 2$, $(s-1)/\alpha \neq$ integer. Let $u$ be the solution of (1), and suppose $u \equiv 0$ for $r > 1$. Then*

$$u(x) = \sum_1^{J(s)} \Lambda_j\{f, h_0, h_1\} v_j(x) + W(x, s-1),$$

*where $\Lambda_j\{f, h_0, h_1\}$, $j = 1, \cdots, J(s)$ are bounded linear functionals on $\mathcal{Y}_N^s(S)$, $v_j$ is given by (14), and, for each $a > 0$, $W \in H^s(S_a)$. There is a constant $C(a) > 0$ such that*

$$||W(\cdot, s-1)||_{H^s(S_a)} \leq C(a)||\{f, h_0, h_1\}||_{\mathcal{Y}_N^s(S)}. \tag{18}$$

Proof. If $s > 2$ and $s \neq$ integer, the result follows from Theorem 1. Suppose $s = k > 2$ is an integer, and $(s-1)/\alpha \neq$ integer. Let $\varepsilon > 0$ be such that $j\alpha \neq (k-\varepsilon, k+\varepsilon)$ for $j = 1, \cdots$. Thus, $J(s-\varepsilon) = J(s+\varepsilon)$, so $v(x, k-\varepsilon-1) = v(k+\varepsilon-1)$, and therefore, $W(x, k-\varepsilon-1) = W(x, k+\varepsilon-1)$. We define $W(x, k-1) = W(x, k-\varepsilon-1)$. From Theorem 1, for each $a > 0$,

$$||W(\cdot, k-\varepsilon-1)||_{H^{k-\varepsilon}(S_a)} \leq C(a)||\{f, h_0, h_1\}||_{\mathcal{Y}_N^{k-\varepsilon}(S)},$$

$$||W(\cdot, k+\varepsilon-1)||_{H^{k+\varepsilon}(S_a)} \leq C(a)||\{f, h_0, h_1\}||_{\mathcal{Y}_N^{k+\varepsilon}(S)}.$$

Hence, by interpolation,

$$||W(\cdot, k-1)||_{H^k(S_a)} \leq C(a)||\{f, h_0, h_1\}||_{\mathcal{Y}_N^k(S)}.$$

If $2 < s \leq 1+\alpha$, $J(s) = 0$, $u = W(\cdot, s-1)$, and the bound for $u$ follows from (13). In the case $s = 2$, the proof is a little different. Let $\{f, h_0, h_1\} \in \mathcal{Y}_N^2(S)$, let $\varepsilon > 0$, and let $\{f^\mu, h_0^\mu, h_1^\mu\} \in \mathcal{Y}_N^{2+\varepsilon}(S)$ be a sequence with $||\{f^\mu - f, h_0^\mu - h_0, h_1^\mu - h_1\}||_{\mathcal{Y}_N^2(S)} \to 0$ as $\mu \to \infty$. Let $u^\mu$ be the solution of (1) with data $\{f^\mu, h_0^\mu, h_1^\mu\}$. Suppose, without loss of generality, that $u^\mu(x) = 0$ for $r > 1$. Suppose $\varepsilon$ is so small that $2+\varepsilon < (J(2)+1)\alpha$, so $J(2+\varepsilon) = J(2)$. Applying Theorem 2 with $s = 2+\varepsilon$, we have the decomposition

$$u^\mu(x) = v^\mu(x, s+\varepsilon-1) + W^\mu(x, s+\varepsilon-1).$$

The singular expansion $v^\mu$ is given by

$$v^\mu(x, s+\varepsilon-1) = \sum_1^{J(2)} \Lambda_j\{f^\mu, h_0^\mu, h_1^\mu\} v_j(x).$$

Since each $\Lambda_j$ is a bounded linear functional on $\mathcal{Y}_N^2(S)$, $v^\mu \to v$ as $\mu \to \infty$ where

$$v(x, 1) = \sum_1^{J(2)} \Lambda_j\{f, h_0, h_1\} r^{j\alpha} \cos j\alpha\theta.$$

Since $\Delta v^\mu = 0$ and $v_n^\mu = 0$ on $\Gamma_0$ and $\Gamma_1$, $\Delta W^\mu = f^\mu$ and $W_n^\mu = h_l^\mu$ on $\Gamma_l$, $l = 0, 1$. From Theorem 2 in the case $s > 2$, $W^\mu \in H^{2+\varepsilon}(S_a)$. Hence we may apply (2.1;26) to $W^\mu - W^\nu$ to obtain $||W^\mu - W^\nu||_{H^2(S)} \to 0$ as $\mu, \nu \to \infty$. Hence $W^\mu$ converges in $H^2(S)$ to a function $W \in H^2(S)$. We therefore obtain the desired decomposition of $u$ in the case $s = 2$. ■

*Calculation of $\Lambda_j$ in the case of distinct poles*

In the confluent case, when $j\alpha = k$ for some integers $j$ and $k$, the linear functionals $\Lambda'_j(h)$ and $\Lambda''_j(f)$ are given by (11) and (12). It is of interest to obtain formulas for these linear functionals in the case $j\alpha \neq$ integer. For this, we must evaluate $\hat{f}(\eta i, \theta)$ for $\eta > -1$ and $\hat{h}_l(\eta i)$ for $\eta > 0$. We have

$$\begin{aligned}\hat{h}_l(\eta i) =& \hat{T}_{l,n-2}(\eta i) + \frac{1}{\sqrt{2\pi}} \int_0^\infty R^*_{l,n-2}(\tau) e^{\eta\tau} d\tau \\ =& \frac{-1}{\sqrt{2\pi}} \sum_0^{n-2} \frac{h_1^{(k)}(0)}{(\eta-k)k!} + \frac{1}{\sqrt{2\pi}} \int_0^1 [h_l(x) - T_{l,n-2}(x)] x^{-\eta-1} dx,\end{aligned}$$

$$
\begin{aligned}
\hat{f}(\eta i,\theta) =&\hat{T}_{n-3}(\eta i,\theta)+\frac{1}{\sqrt{2\pi}}\int_0^\infty R_n^*(\tau,\theta)e^{\eta\tau}d\tau\\
=&\frac{-1}{\sqrt{2\pi}}\sum_{0\le k+m\le n-3}\frac{\cos^k\theta\sin^m\theta}{k!m!(\eta-(k+m))}[D_{x_1}^kD_{x_2}^m f(0,0)]\\
&+\frac{1}{\sqrt{2\pi}}\int_0^\infty[f(r\cos\theta,r\sin\theta)-T_{n-3}(r\cos\theta,r\sin\theta)]r^{-\eta-1}dr.
\end{aligned}
$$

Hence from (8b),

$$
(19a)\qquad \Lambda_j'(h_l)=-\frac{1}{j\pi}\sum_0^{n-2}\frac{h_l^{(k)}(0)}{(j\alpha-k-1)k!}+\frac{1}{j\pi}\int_0^1[h_l(x)-T_{l,n-2}(x)]x^{-j\alpha}dx,\ j\alpha\ne\text{integer},
$$

$$
(19b)\qquad
\begin{aligned}
\Lambda_j''(f)=&-\frac{1}{j\pi}\sum_{0\le k+m\le n-3}\frac{A_{k,m}(\omega)}{k!m!(j\alpha-(k+m+2))}[D_{x_1}^kD_{x_2}^m f(0,0)]\\
&+\frac{1}{j\pi}\int\int_{S_1}[f(x)-T_{n-3}(x)]r^{-j\alpha}\cos j\alpha\theta dx,\ j\alpha\ne\text{integer},
\end{aligned}
$$

where

$$
A_{k,m}(\omega)=\int_0^\omega\cos^k\varphi\sin^m\varphi\cos j\alpha\varphi d\varphi.
$$

From the construction, if $j\alpha\ne$ integer, $\{f,h_0,h_1\}\to\Lambda_j'(h_l)$ and $\{f,h_0,h_1\}\to\Lambda_j''(f)$ are bounded linear functionals on $\mathcal{Y}_N^s(S)$. We verify this directly. Since $h_l\in H^{s-3/2}(R_+)$ and $s=n+\sigma$ with $0<\sigma<1$, $h_l$ is $(n-2)$-times continuously differentiable on $R_+$, so $h_l\to h_l^{(k)}$ is a bounded linear functional on $H^{s-3/2}(R_+)$ if $k\le n-2$. Also, from Theorem I.2;6,

$$
\int_0^1 x^{-2(s-3/2)}|h_l-T_{l,n-2}|^2dx\le C||h_l||_{s-3/2}^2.
$$

Since $j\alpha<s-1$,

$$
\begin{aligned}
|\int_0^1[h_l(x)-T_{l,n-2}(x)]x^{-j\alpha}dx|\le&\int_0^1 x^{-(s-3/2)}|h_l(x)-T_{l,n-2}(x)|\cdot x^{s-3/2-j\alpha}dx\\
\le&\big(\int_0^1 x^{2(s-j\alpha-3/2)}dx\big)^{1/2}||h_l||_{s-3/2}\\
\le&C||h_l||_{s-3/2}.
\end{aligned}
$$

Hence $\Lambda_j'(h_l)$ is a bounded linear functional on $H^{s-3/2}(R_+)$, and so is a bounded linear functional on $\mathcal{Y}_N^s(S)$. Similarly, since $f\in H^{s-2}(S)$, $f$ has $n-3$ continuous derivatives, provided $s>3$. Hence $f\to D_{x_1}^kD_{x_2}^mf(0,0)$ is a bounded linear functional on $\mathcal{Y}_N^s(S)$ for $k+m\le n-3$, provided $s>3$. Also, from Theorem I.4;6,

$$
\int\int_{S_1}r^{-2(s-2)}[f-T_{n-3}]^2dx\le C||f||_{s-2}^2.
$$

Since $j\alpha<s-1$,

$$
\begin{aligned}
|\int\int_{S_1}[f-T_{n-3}]r^{-j\alpha}\sin j\alpha\varphi dx|\le&\int\int_{S_1}r^{-(s-2)}|f-T_{n-3}|\cdot r^{-j\alpha+s-2}dx\\
\le&C\big(\int\int_{S_1}r^{2(-j\alpha+s-2)}dx\big)^{1/2}||f||_{s-2}\\
\le&C||f||_{s-2}.
\end{aligned}
$$

Hence $\Lambda_j''(f)$ is a bounded linear functional on $\mathcal{Y}_N^s(S)$.

From the formulas, it is seen that $\Lambda_j\{f,h_0,h_1\}$, and hence, $v_{j,0}$, becomes singular if $j\alpha\to$ integer. This does not contradict the assertion in Theorem 1 that $v$ is a continuous function of $\omega$, because, as will be seen, the singularity in $v_{j,0}$ that appears when $j\alpha\to$ integer is cancelled by a singularity of opposite sign in $v_{0,k}$.