2.2.3 Solution formula

The formula (2.2;3) gives a formula for the solution u of the problem (2.2;1a,b,c) in terms of the Kondrat'yev transform $\hat{u}$. Here, the inverse transform is taken to derive a formula for u.

Equation (2.2;3) may be written $\hat{u} = \hat{u}_i + \hat{u}_0 + \hat{u}_1$ where

$$\begin{aligned}\hat{u}_i(\zeta,\theta) = &\int_0^\theta \hat{f}(\zeta-2i,\varphi)\frac{\sinh\zeta(\omega-\theta)\sinh\zeta\varphi}{\zeta\sinh\zeta\omega}d\varphi \\ &+\int_\theta^\omega \hat{f}(\zeta-2i,\varphi)\frac{\sinh\zeta\theta\sinh\zeta(\omega-\varphi)}{\zeta\sinh\zeta\omega}d\varphi, \\ \hat{u}_0(\zeta,\theta) =&\hat{g}_0(\zeta)\frac{\sinh\zeta(\omega-\theta)}{\sinh\zeta\omega}, \\ \hat{u}_1(\zeta,\theta) =&\hat{g}_1(\zeta)\frac{\sinh\zeta\theta}{\sinh\zeta\omega}.\end{aligned}$$

It is shown in Section I.2.2 that this formula is valid for $\Im\zeta = \eta \leq 0$ if $f \in H^1(S)$ and $\{g_0, g_1\} \in H^{-1/2}(\Gamma)$. With more regularity on the data, the formula is valid for $\eta < \alpha$.

Since $u_l^*(\tau,\theta)$ is the inverse Fourier transform of $\hat{u}_l(\xi,\theta)$, $l = 0, 1, i$, one has

$$\begin{aligned}u_0^*(\tau,\theta) =&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{u}_0(\xi,\theta)e^{i\xi\tau}d\xi \\ =&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{g}_0(\xi)e^{i\xi\tau}\frac{\sinh\xi(\omega-\theta)}{\sinh\xi\omega}d\xi.\end{aligned}$$

Since r and τ are related by the formula $r = e^{-\tau}$, this gives

$$u_0(r\cos\theta, r\sin\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{g}_0(\xi)r^{-i\xi}\frac{\sinh\xi(\omega-\theta)}{\sinh\xi\omega}d\xi.$$

From Section I.1,

$$\hat{g}_0(\xi) = \frac{1}{\sqrt{2\pi}}\int_0^\infty s^{i\xi-1}g_0(s)ds.$$

We therefore obtain

$$\begin{aligned}u_0(r\cos\theta, r\sin\theta) &= \frac{1}{2\pi}\int_{\xi=\infty}^\infty\int_{s=0}^\infty g_0(s)s^{i\xi-1}r^{-i\xi}\frac{\sinh\xi(\omega-\theta)}{\sinh\xi\omega}dsd\xi \\ &= \int_0^\infty g_0(s)G_0(r,s,\theta)ds,\end{aligned}$$

where

$$\begin{aligned}G_0(r,s,\theta) &= \frac{1}{2\pi}\int_{-\infty}^\infty s^{i\xi-1}r^{-i\xi}\frac{\sinh\xi(\omega-\theta)}{\sinh\xi\omega}d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty s^{i\xi-1}r^{-i\xi}\frac{e^{(\omega-\theta)\xi}-e^{-(\omega-\theta)\xi}}{e^{\omega\xi}-e^{-\omega\xi}}d\xi \\ &= \frac{1}{\pi s}\int_0^\infty \cos\left(\xi\ln(r/x)\right)\frac{e^{(\omega-\theta)\xi}-e^{-(\omega-\theta)\xi}}{e^{\omega\xi}-e^{-\omega\xi}}d\xi.\end{aligned} \tag{1}$$

Similarly,

$$u_1(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_{\xi=\infty}^{\infty}\int_{s=0}^{\infty} g_1(s)s^{i\xi-1}r^{-i\xi}\frac{\sinh\xi\theta}{\sinh\xi\omega}dsd\xi$$
$$= \int_0^\infty g_1(s)G_1(r,s,\theta)ds,$$

where

$$G_1(r,s,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s^{i\xi-1}r^{-i\xi}\frac{\sinh\xi\theta}{\sinh\xi\omega}d\xi$$
$$(2) \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} s^{i\xi-1}r^{-i\xi}\frac{e^{\theta\xi}-e^{-\theta\xi}}{e^{\omega\xi}-e^{-\omega\xi}}d\xi$$
$$= \frac{1}{\pi s}\int_0^\infty \cos\left(\xi\ln(r/x)\right)\frac{e^{\theta\xi}-e^{-\theta\xi}}{e^{\omega\xi}-e^{-\omega\xi}}d\xi.$$

Finally,

$$u_i(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_0^\theta\int_{\xi=\infty}^{\infty}\int_{s=0}^{\infty} f(s\cos\varphi, s\sin\varphi)s^{i\xi}r^{-i\xi}\frac{k(\xi,\omega-\theta)k(\xi,\varphi)}{k(\xi,\omega)}sdsd\xi d\varphi$$
$$+ \frac{1}{2\pi}\int_\theta^\omega\int_{\xi=\infty}^{\infty}\int_{s=0}^{\infty} f(s\cos\varphi, s\sin\varphi)s^{i\xi}r^{-i\xi}\frac{k(\xi,\theta)k(\xi,\omega-\varphi)}{k(\xi,\omega)}sdsd\xi d\varphi$$
$$= \int_0^\omega\int_0^\infty f(s\cos\varphi, s\sin\varphi)G_i(r,\theta,s,\varphi)sdsd\varphi,$$

where

$$G_i(r,\theta,s,\varphi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s^{i\xi}r^{-i\xi}\frac{\sinh\zeta(\omega-\theta)\sinh\zeta\varphi}{\zeta\sinh\zeta\omega}d\xi$$
$$= \frac{1}{2\pi}\int_0^\infty \cos\left(\xi\ln(r/x)\right)\frac{[e^{(\omega-\theta)\xi}-e^{-(\omega-\theta)\xi}][e^{\varphi\xi}-e^{-\varphi\xi}]}{\xi[e^{\omega\xi}-e^{-\omega\xi}]}d\xi \text{ for } 0<\varphi<\theta,$$
$$(3) \qquad G_i(r,\theta,s,\varphi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s^{i\xi}r^{-i\xi}\frac{\sinh\zeta\theta\sinh\zeta(\omega-\varphi)}{\zeta\sinh\zeta\omega}d\xi$$
$$= \frac{1}{2\pi}\int_0^\infty \cos\left(\xi\ln(r/x)\right)\frac{[e^{\theta\xi}-e^{-\theta\xi}][e^{(\omega-\varphi)\xi}-e^{-(\omega-\varphi)\xi}]}{\xi[e^{\omega\xi}-e^{-\omega\xi}]}d\xi \text{ for } \theta<\varphi<\omega.$$

Summarizing the above formulas, the solution u of (2.2;1a,b,c) is given by

$$(4) \qquad u(r\cos\theta, r\sin\theta) = \int_0^\infty g_0(s)G_0(r,s,\theta)ds + \int_0^\infty g_1(s)G_1(r,s,\theta)ds$$
$$+ \int_0^\omega\int_0^\infty f(s\cos\varphi, s\sin\varphi)G_i(r,\theta,s,\varphi)sdsd\varphi,$$

where the functions G_0, G_1, and G_i are given respectively by (1), (2), and (3).